SN spaces, \( r_L \)-density and maximal monotonicity

by

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Abstract

\( r_L \)-density is a concept that can be applied to subsets of \( E \times E^* \), where \( E \) is a nonzero real Banach space. We start our discussion of it in the more general situation of subsets of SN spaces, where the notation is more concise. In the \( E \times E^* \) case, every closed \( r_L \)-dense monotone set is maximally monotone, but there exist maximally monotone sets that are not \( r_L \)-dense. The graph of the subdifferential of a proper, convex lower semicontinuous function on \( E \) is \( r_L \)-dense. The graphs of certain subdifferentials of certain nonconvex functions are also \( r_L \)-dense. (This follows from joint work with Xianfu Wang.) The closed monotone and \( r_L \)-dense sets have a number of very desirable properties, including a sum theorem under both natural and unnatural constraint conditions, so \( r_L \)-density satisfies the ideal calculus rules. We also give a generalization of the Brezis–Browder theorem on linear relations.

Downloads

You can download files containing complete proofs and many references from the web. I will give you the link at the end of the talk.
**SN spaces**

Symmetric linear maps, the associated quadratic form $q_L$, and SN spaces. 
$L$–positive sets. 
The function $r_L$, $r_L$–density and maximality. 
The function $s_L$, a criterion for the $r_L$–density of certain sets. 
Polar subspaces. 
The function $\Phi_A$ and the Fitzpatrick extension. 

**The $E \times E^*$ case**

The tail. 
Subdifferentials of convex and non convex functions. 
A negative alignment criterion for $r_L$–density. 
The convexity of $\overline{D(S)}$ and $\overline{R(S)}$. 
Type (ANA). 
Sum theorems and subdifferential perturbation theorems. 
**Strong** maximality, type (FP) and type (NI). 
Generalizations of the Brezis–Browder theorem.
— SN spaces, $\mathcal{R}_L$–density and maximal monotonicity—

### Symmetric linear maps

Let $B$ be a nonzero real Banach space. A linear map $L: B \to B^*$ is **symmetric** if, $\forall \, b, c \in B$, $\langle b, Lc \rangle = \langle c, Lb \rangle$. The quadratic form $q_L$ on $B$ is defined by

$$q_L(b) := \frac{1}{2} \langle b, Lb \rangle.$$

- We have the parallelogram law:

  $$b, c \in B \implies \frac{1}{2} q_L(b - c) + \frac{1}{2} q_L(b + c) = q_L(b) + q_L(c).$$

### Definition of SN space

$B$ (more precisely, $(B, L)$) is a **symmetric nonexpansion space (SN space)** if $B$ is a nonzero real Banach space and $L: B \to B^*$ is a symmetric nonexpansive linear map from $B$ into $B^*$.

### Examples of SN spaces

(a) If $B$ is a Hilbert space then $B$ is an **SN space** with $Lc := c$. Then $q_L(b) = \frac{1}{2} \|b\|^2$.

(b) If $B$ is a Hilbert space then $B$ is an **SN space** with $Lc := -c$. Then $q_L(b) = -\frac{1}{2} \|b\|^2$.

(c) $\mathbb{R}^3$ is an **SN space** with $L(c_1, c_2, c_3) := (c_2, c_1, c_3)$. Then $q_L(b_1, b_2, b_3) = b_1 b_2 + \frac{1}{2} b_3^2$.

(d) $\mathbb{R}^3$ is **not** an **SN space** with $L(c_1, c_2, c_3) := (c_2, c_3, c_1)$ since

$$\langle (0, 1, 0), L(1, 0, 0) \rangle = 0 \quad \text{but} \quad \langle (1, 0, 0), L(0, 1, 0) \rangle = 1.$$
Definition of SN space

A (more precisely, \((B, L)\)) is a \textit{symmetric nonexpansion space (SN space)} if \(B\) is a nonzero real Banach space and \(L: B \to B^*\) is a symmetric nonexpansive linear map from \(B\) into \(B^*\).

An example of an SN space motivated by monotonicity

(e) Let \(E\) be a nonzero Banach space and \(B := E \times E^*\) under the norm 
\[
\| (x, x^*) \| := \sqrt{\| x \|^2 + \| x^* \|^2}.
\]

Let \((E \times E^*, \| \cdot \|) = (E^* \times E^{**}, \| \cdot \|)\), with \(\| (y^*, y^{**}) \| := \sqrt{\| y^* \|^2 + \| y^{**} \|^2}\) and 
\[
\langle (x, x^*), (y^*, y^{**}) \rangle := \langle x, y^* \rangle + \langle x^*, y^{**} \rangle.
\]

\(\forall (y, y^*) \in B,\) let 
\[
L(y, y^*) := (y^*, \hat{y}),
\]

where \(\hat{y}\) is the canonical image of \(y\) in \(E^{**}\). Since 
\[
\langle (x, x^*), L(y, y^*) \rangle = \langle x, y^* \rangle + \langle x^*, \hat{y} \rangle = \langle y, x^* \rangle + \langle y^*, \hat{x} \rangle = \langle (y, y^*), L(x, x^*) \rangle,
\]

\(B\) is an SN space, and 
\[
q_L(x, x^*) = \langle x, x^* \rangle.
\]

Any finite dimensional SN space of this form must have \textit{even} dimension. Thus odd dimensional cases of the examples considered on the previous slide \texttt{cannot} be of this form. In contrast to the three examples on the previous slide, if \(E\) is not reflexive then \(L\) is not surjective.

\(\bullet\) \(E\) will always be a nonzero Banach space and so, with \(L\) as defined above, \((E \times E^*, L)\) is an SN space. \(B\) will be an SN space.
Definition of $L$–positive set

Let $A \subset B$. We say that $A$ is $L$–positive if $A \neq \emptyset$ and

$$b, c \in A \implies q_L(b - c) \geq 0.$$  

Examples of $L$–positive sets

(a) $B$ is a Hilbert space with $Lc := c$: every nonempty subset of $B$ is $L$–positive.
(b) $B$ is a Hilbert space with $Lc := -c$: the $L$–positive subsets of $B$ are the singletons.
(e) $E$ is a nonzero Banach space, $B := E \times E^*$ and, $L(x, x^*) := (x^*, \hat{x})$. Let $\emptyset \neq A \subset B$. Then $A$ is $L$–positive when

$$(x, x^*), (y, y^*) \in A \implies \langle x - y, x^* - y^* \rangle \geq 0.$$  

That is to say,

$$A \text{ is } L\text{-positive} \iff A \text{ is a monotone subset of } E \times E^*.$$  

General notation

- Let $X$ be a vector space and $f: X \mapsto ]-\infty, \infty]$. Then $\text{dom } f := \{x \in X: f(x) \in \mathbb{R}\}$.
- $f$ is proper if $\text{dom } f \neq \emptyset$.
- $\mathcal{PC}(X)$ is the set of all proper convex functions $f: X \mapsto ]-\infty, \infty]$.
- If $X$ is a Banach space, $\mathcal{PC}_{LSC}(X) := \{f \in \mathcal{PC}(X): f \text{ is lower semicontinuous}\}$.
- If $f, g: X \to [-\infty, \infty]$, then $\{X | f = g\}$ is the “equality set” $\{x \in X | f(x) = g(x)\}$. 
SN spaces, $r_L$–density and maximal monotonicity

SN space notation

- If $(B, L)$ is a Banach SN space, $\mathcal{PC}_q(B) := \{f \in \mathcal{PC}(B): f \geq q_L \text{ on } B\}$.
- If $(B, L)$ is a Banach SN space, $\mathcal{PCLSC}_q(B) := \{f \in \mathcal{PCLSC}(B): f \geq q_L \text{ on } B\}$.

The $L$–positive set given by a convex function

If $f \in \mathcal{PC}_q(B)$ and $\{B|f = q_L\} \neq \emptyset$ then $\{B|f = q_L\}$ is an $L$–positive subset of $B$.

Proof. Let $b, c \in B$, $f(b) = q_L(b)$ and $f(c) = q_L(c)$. Then, from the parallelogram law, the quadraticity of $q_L$, and the convexity of $f$,

$$\frac{1}{2}q_L(b - c) = q_L(b) + q_L(c) - \frac{1}{2}q_L(b + c) = q_L(b) + q_L(c) - 2q_L\left(\frac{1}{2}(b + c)\right) \geq f(b) + f(c) - 2f\left(\frac{1}{2}(b + c)\right) \geq 0.$$

$\blacksquare$
Definition of the function $r_L$

Let $b \in B$. Then $r_L(b) := \frac{1}{2} \|b\|^2 + q_L(b)$.

- Since $\|L\| \leq 1$, $b \in B \implies \frac{1}{2} \|b\|^2 \geq -q_L(b)$. Consequently
  \[ b \in B \implies r_L(b) \geq 0. \]

Examples of the function $r_L$

(a) $B$ is a Hilbert space with $Lb := b$. Then
  \[ r_L(b) = \|b\|^2. \]

(b) $B$ is a Hilbert space with $Lb := -b$. Then
  \[ r_L(b) = 0. \]

(e) $E$ is a nonzero Banach space, $B := E \times E^*$ and, $L(x, x^*) := (x^*, \hat{x})$. Then
  \[ r_L(x, x^*) = \frac{1}{2} \|x\|^2 + \langle x, x^* \rangle + \frac{1}{2} \|x^*\|^2. \]

Definition of $r_L$–density

Let $A \subset B$. Then $A$ is $r_L$–dense if
\[
\forall \ b \in B, \quad \inf r_L(A - b) = 0.
\]

- This means: $\forall \ b \in B$ and $\varepsilon > 0 \ \exists \ a \in A$ such that $r_L(a - b) < \varepsilon$.  

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**Definition of $r_L$–density**

Let $A \subset B$. Then $A$ is $r_L$–dense if

$$\forall \ b \in B, \ \inf r_L(A - b) = 0.$$ 

- This means: $\forall \ b \in B$ and $\varepsilon > 0 \ \exists \ a \in A$ such that $r_L(a - b) < \varepsilon$.

**Theorem: $r_L$–density implies maximality**

Let $A$ be a closed $r_L$–dense $L$–positive subset of $B$. Then $A$ is maximally $L$–positive (in the obvious sense).

**Proof.** Suppose that $b \in B$ and $A \cup \{b\}$ is $L$–positive. Let $\varepsilon > 0$. By hypothesis,

$$\exists \ a \in A \text{ such that } r_L(a - b) < \varepsilon.$$ 

Thus

$$\frac{1}{2}\|a - b\|^2 + q_L(a - b) < \varepsilon.$$ 

Since $A \cup \{b\}$ is $L$–positive, $q_L(a - b) \geq 0$, and so

$$\frac{1}{2}\|a - b\|^2 \leq \varepsilon.$$ 

However, $A$ is closed. Thus, letting $\varepsilon \to 0$, we have

$$b \in A.$$ 

- As we will see later, the converse of the above result is false. There are maximally $L$–positive subsets of $E \times E^*$ that are not $r_L$–dense.
Definition of the function $s_L$

Let $b^* \in B^*$. We define $s_L(b^*) \in [-\infty, \infty]$ by

$$s_L(b^*) := \sup_{b \in B} \left[ \langle b, b^* \rangle - q_L(b) - \frac{1}{2} \| Lb - b^* \|^2 \right].$$

The reason for this strange definition will appear on the next slide.

Examples of the function $s_L$

(a) $B$ is a Hilbert space with $Lb := b$. Then

$$s_L(b^*) = \frac{1}{2} \| b^* \|^2.$$

(e) $E$ is a nonzero Banach space, $B := E \times E^*$ and, $L(x, x^*) := (x^*, \widehat{x})$. Then

$$s_L(x^*, x^{**}) = \sup_{(y, y^*) \in E \times E^*} \left[ \langle y, x^* \rangle + \langle y^*, x^{**} \rangle - \langle y, y^* \rangle - \frac{1}{2} \| y^* - x^* \|^2 - \frac{1}{2} \| \widehat{y} - x^{**} \|^2 \right].$$

Mercifully, this simplifies to the formula

$$\forall (x^*, x^{**}) \in E^* \times E^{**}, \quad s_L(x^*, x^{**}) = \langle x^*, x^{**} \rangle. \quad (\heartsuit)$$
Criterion for \( \{ B | f = q_L \} \) to be \( r_L \)-dense in \( B \)

Let \( f \in \mathcal{PCLSC}_q(B) \). Then:

\( \{ B | f = q_L \} \) is \( r_L \)-dense in \( B \) \iff \( f^* \geq s_L \) on \( B^* \).

**Proof.** One can prove that both conditions above are equivalent to

\[
\forall \ c \in B, \ \inf_{b \in B} \left[ (f - q_L)(b) + r_L(b - c) \right] \leq 0.
\]

The proof of the equivalence with the left hand condition uses a completeness argument. The proof of the equivalence with the right hand condition uses Rockafellar’s version of the Fenchel duality theorem on the two functions \( f \) and \( g_c \), where

\[
g_c(b) := -q_L(b) + r_L(b - c) = -\langle b, Lc \rangle + q_L(c) + \frac{1}{2} \| b - c \|^2,
\]

which is continuous and convex. The definition of \( s_L \) was obtained by working backwards from this proof. For more details, see the material on the web. \( \square \)
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Polar subspace

• If $Y$ is a linear subspace of a Banach space $X$, $Y^0$ is the “polar subspace of $Y$”, that is to say the linear subspace $\{ x^* \in X^* : \langle Y, x^* \rangle = \{0\} \}$ of $X^*$.

<table>
<thead>
<tr>
<th>Theorem on the $r_L$-density of linear subspaces</th>
</tr>
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<tbody>
<tr>
<td>Let $(B, L)$ be an SN space and $A$ be a closed linear $L$–positive subspace of $B$. Then</td>
</tr>
<tr>
<td>$A$ is $r_L$-dense in $B$ $\iff$ $\sup L(A^0) \leq 0$.</td>
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</tbody>
</table>

Comment. We will see later that the above result leads to a generalization of the Brezis–Browder theorem on the monotonicity of the adjoint relation.
— SN spaces, $r_L$–density and maximal monotonicity—

• We have shown how $f \in \mathcal{PC}_q(B)$ leads to the $L$–positive set, $\{B|f = q_L\}$.
• We now consider the converse problem: given an $L$–positive set, $A$, we show how to obtain a convex function, $\Phi_A$, on $B$.

**A convex function given by an $L$–positive set**

Let $A$ be an $L$–positive subset of $B$. We define $\Phi_A: B \mapsto ]-\infty, \infty]$ by

$$\Phi_A(b) := \sup_A [Lb - q_L] = \sup_{a \in A} [\langle a, bLb \rangle - q_L(a)].$$

**Nice property of $\Phi_A$**

Let $A$ be a maximally $L$–positive subset of $B$. Then

$$\Phi_A \in \mathcal{PC}\mathcal{L}\mathcal{S}\mathcal{C}_q(B) \quad \text{and} \quad \{B|\Phi_A = q_L\} = A.$$ 

**Criterion for $\{B|f = q_L\}$ to be $r_L$–dense in $B$**

Let $f \in \mathcal{PC}\mathcal{L}\mathcal{S}\mathcal{C}_q(B)$. Then:

$$\{B|f = q_L\} \text{ is } r_L\text{–dense in } B \iff f^* \geq s_L \text{ on } B^*.$$ 

**Theorem on the $r_L$–density of an $L$–positive set**

Let $A$ be a closed $L$–positive subset of $B$. Then $A$ is $r_L$–dense in $B$ if, and only if, $A$ is maximally $L$–positive and $\Phi_A^* \geq s_L$ on $B^*$.

**Proof.** Immediate from the two results above with $f := \Phi_A$. 

\[\square\]
— SN spaces, \( r_L \)-density and maximal monotonicity —

**Theorem on the \( r_L \)-density of an \( L \)-positive set**

A be a closed \( L \)-positive subset of \( B \). Then \( A \) is \( r_L \)-dense in \( B \) if, and only if, \( A \) is maximally \( L \)-positive and \( \Phi_A^* \geq s_L \) on \( B^* \).

**Definition of the Fitzpatrick extension**

Let \( A \) be a closed, \( r_L \)-dense \( L \)-positive subset of \( B \) and \( s_L \circ L = q_L \). The Fitzpatrick extension of \( A \) is the set \( A^F := \{ B^* | \Phi_A^* = s_L \} \).

- The reason that we use the word “extension” is that frequently \( L^{-1} A^F = A \).

**Remember \( (\star) \)**

For all \( (x^*, x^{**}) \in E^* \times E^{**} \),

\[
 s_L(x^*, x^{**}) = \langle x^*, x^{**} \rangle.
\]  

(\( \star \))

**Multifunction notation**

- For the rest of this paper, we will suppose that \( B = E \times E^* \), as in case (e).
- If \( S: E \rightrightarrows E^* \) let \( D(S) := \{ x \in E: Sx \neq \emptyset \} \) and \( R(S) := \bigcup_{x \in E} Sx \).
- If \( S: E \rightrightarrows E^* \) let \( \varphi_S := \Phi_G(S) \). \( \varphi_S \) is known as the “Fitzpatrick function” of \( S \).
- If \( S: E \rightrightarrows E^* \) we say that \( S \) is closed if \( G(S) \) is a closed subset of \( E \times E^* \).
- If \( S: E \rightrightarrows E^* \) we say that \( S \) is \( r_L \)-dense if \( G(S) \) is an \( r_L \)-dense subset of \( E \times E^* \).
- If \( S: E \rightrightarrows E^* \) is closed, monotone and \( r_L \)-dense, we define \( S^F: E^* \rightrightarrows E^{**} \), by \( G(S^F) := G(S)^F \). Using \( (\star) \), \( x^{**} \in S^F(x^*) \iff \varphi_S^*(x^*, x^{**}) = s_L(x^*, x^{**}) = \langle x^*, x^{**} \rangle \). We will call \( S^F \) the Fitzpatrick extension of \( S \).
The tail...

Let $E = \ell^1$, and define $T: \ell^1 \to \ell^\infty = E^*$ by

$$(Tx)_n = \sum_{k=n}^\infty x_k.$$  

$T$ is the “tail” operator. Then $T$ is maximally monotone but not $r_L$–dense.

**Proof.** It is well known that $T$ is maximally monotone. Let

$$e^* := (1, 1, \ldots) \in \ell_1^* = \ell_\infty.$$  

Let $x \in \ell_1$, and write \(\sigma = \langle x, e^* \rangle = \sum_{n \geq 1} x_n.\) Clearly, \(\|x\| \geq \sigma.\) Since $Tx \in c_0$, we also have \(\|Tx - e^*\| = \sup_n \| (Tx)_n - 1 \| \geq \lim_n \| (Tx)_n - 1 \| = 1.\) Thus

$$\langle x, Tx \rangle = \sum_{n \geq 1} x_n \sum_{k \geq n} x_k = \sum_{n \geq 1} x_n^2 + \sum_{n \geq 1} \sum_{k > n} x_n x_k \geq \frac{1}{2} \sum_{n \geq 1} x_n^2 + \sum_{n \geq 1} \sum_{k > n} x_n x_k = \frac{1}{2} \sigma^2.$$  

It follows that

$$r_L((x, Tx) - (0, e^*)) = \frac{1}{2} \|x\|^2 + \frac{1}{2} \|Tx - e^*\|^2 + \langle x, Tx - e^* \rangle \geq \frac{1}{2} \sigma^2 + \frac{1}{2} + \langle x, Tx \rangle - \sigma \geq \frac{1}{2} \sigma^2 + \frac{1}{2} + \frac{1}{2} \sigma^2 - \sigma = \sigma^2 + \frac{1}{2} - \sigma \geq \frac{1}{4}.$$  

Consequently, $T$ is not $r_L$–dense. □
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**Criterion for** $\{B|f = q_L\}$ **to be** $r_L$–**dense** in $B$

Let $f \in \mathcal{PCLSC}_q(B)$. Then:

$\{B|f = q_L\}$ is $r_L$–dense in $B$ $\iff$ $f^* \geq s_L$ on $B^*$.

**Remember (☆)

For all $(x^*,x^{**}) \in E^* \times E^{**}$, $s_L(x^*,x^{**}) = \langle x^*,x^{**} \rangle$. (☆)**

**Theorem on subdifferentials**

Let $k \in \mathcal{PCLSC}(E)$. Then $\partial k$ is $r_L$–dense in $E \times E^*$.

**Proof.** Define $f \in \mathcal{PCLSC}(E \times E^*)$ by $f(x,x^*) := k(x) + k^*(x^*)$. From the Fenchel–Young inequality,

$$f(x,x^*) \geq \langle x,x^* \rangle = q_L(x,x^*),$$

so $f \in \mathcal{PCLSC}_q(E \times E^*)$. By direct computation, $\forall (x^*,x^{**}) \in E^* \times E^{**}$,

$$f^*(x^*,x^{**}) := k^*(x^*) + k^{**}(x^{**}).$$

From the Fenchel–Young inequality again and (☆),

$$f^*(x^*,x^{**}) \geq \langle x^*,x^{**} \rangle = s_L(x^*,x^{**}).$$

From the criterion above, $\{E \times E^*|f = q_L\}$ is $r_L$–dense in $E \times E^*$.

But this set is exactly $G(\partial k)$. □

**Comment.** Since $G(\partial k)$ is closed, this result is a strict generalization of Rockafellar’s theorem on the maximal monotonicity of subdifferentials.
A brief digression to non convex subdifferentials and non monotone sets
(joint work with Xianfu Wang)

Weak subdifferentials

A weak subdifferential, \( \partial_w \), is a rule that associates with each proper lower semicontinuous function \( f: E \to ]-\infty, \infty] \) a multifunction \( \partial_w f: E \rightrightarrows E^* \) such that,

- \( 0 \in \partial_w f(x) \) if \( f \) attains a strict global minimum at \( x \).
- \( \partial_w (f+h)(x) \subseteq \partial_w f(x) + \partial h(x) \) whenever \( h \) is a continuous convex real function on \( E \) (here \( \partial h \) is the subdifferential of \( h \) of convex analysis).

Comment. The abstract subdifferential introduced by Thibault and Zagrodny gives a weak subdifferential. This implies that a number of other subdifferentials that have been introduced over the years also give weak subdifferentials. In particular, the Clarke-Rockafellar subdifferential is a weak subdifferential. Also, Mordukhovich’s limiting subdifferential is a weak subdifferential if we confine our attention to Asplund spaces.

The \( r_L \)-density of weak subdifferentials

Let \( \partial_w \) be a weak subdifferential and \( k: E \to \mathbb{R} \) be proper, lower semicontinuous and bounded below by a continuous affine functional. Then

\[ \partial_w k \text{ is } r_L\text{-dense in } E \times E^*. \]

Comment. Of course, \( \partial_w k \) is not necessarily monotone if \( k \) is not convex.

For the rest of this talk, we return to the monotone case.
Sufficient conditions for $r_L$-density

Let $S : E \rightrightarrows E^*$ be maximally monotone.

- If $R(S) = E^*$ then $S$ is $r_L$-dense.
- If $E$ is reflexive then $S$ is $r_L$-dense.

- If $X$ and $Y$ are nonempty sets, define $\pi_1 : X \times Y \mapsto X$ and $\pi_2 : X \times Y \mapsto Y$ by $\pi_1(x, y) := x$ and $\pi_2(x, y) := y$.

Theorem on domain and range

Let $S : E \rightrightarrows E^*$ be closed, monotone and $r_L$-dense. Then

$$\overline{D(S)} = \overline{\pi_1(\text{dom } \varphi_S)} \quad \text{and} \quad \overline{R(S)} = \overline{\pi_2(\text{dom } \varphi_S)}.$$

Consequently, $\overline{D(S)}$ and $\overline{R(S)}$ are convex.

Comments. Gossez gave an example of a maximally monotone multifunction for which $\overline{R(S)}$ is not convex.

An example of a maximally monotone multifunction for which $\overline{D(S)}$ is not convex would lead to a counterexample for the sum problem!
A negative alignment criterion for $r_L$–density

Let $S: E \Rightarrow E^*$ be closed and monotone. Then

$S$ is $r_L$–dense

\[
\forall (w, w^*) \in E \times E^*, \exists \tau \geq 0 \text{ and a sequence } \{(s_n, s_n^*)\}_{n \geq 1} \text{ in } G(S) \text{ such that }
\]

\[
\lim_{n \to \infty} \|s_n - w\| = \tau, \quad \lim_{n \to \infty} \|s_n^* - w^*\| = \tau \quad \text{and} \quad \lim_{n \to \infty} \langle s_n - w, s_n^* - w^* \rangle = -\tau^2.
\]

Comments. (⇑) is obvious. (⇓) is more delicate because the boundedness of the sequence is not obvious. For more details, see the material on the web.
Definition of type (ANA)

Let $S : E \rightrightarrows E^*$ be maximally monotone. Then $S$ is of type (ANA) if, whenever $(w, w^*) \in E \times E^* \setminus G(S)$, there exists $(s, s^*) \in G(S)$ such that $s \neq w$, $s^* \neq w^*$, and

$$\frac{\langle s - w, s^* - w^* \rangle}{\|s - w\| \|s^* - w^*\|}$$

is as near as we please to $-1$.

- This was a property originally proved for subdifferentials.
- We do not have an example of a maximally monotone multifunction that is not of type (ANA).

Theorem on type (ANA)

Let $S : E \rightrightarrows E^*$ be closed, monotone and $r_L$–dense. Then $S$ is maximally monotone of type (ANA).

Proof. Immediate from the properties of $\tau$ on the previous slide. \qed
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Partial episums

• Let $X$ and $Y$ be nonzero Banach spaces and $f, g \in \mathcal{PCLSC}(X \times Y)$. Then we define the functions $f \oplus_2 g$ and $f \oplus_1 g$ by

$$(f \oplus_2 g)(x, y) := \inf \{ f(x, y - \eta) + g(x, \eta) : \eta \in Y \}$$

and

$$(f \oplus_1 g)(x, y) := \inf \{ f(x - \xi, y) + g(\xi, y) : \xi \in X \}.$$  

• We substitute the symbol $\oplus_1^e$ for $\oplus_2$ and $\oplus_1^e$ for $\oplus_1$ if the infimum is exact, that is to say, can be replaced by a minimum.

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A bivariate version of the Fenchel duality theorem

Let $f, g \in \mathcal{PCLSC}(X \times Y), f \oplus_2 g \in \mathcal{PC}(X \times Y)$ and

$$\bigcup_{\lambda > 0} \lambda[\pi_1 \text{dom } f - \pi_1 \text{dom } g]$$

be a closed subspace of $X$.

Then

$$(f \oplus_2 g)^* = f^* \oplus_1^e g^*$$
on $X^* \times Y^*$. 

• If $S, T : E \rightrightarrows E^*$ then, $\forall x \in E$, $(S + T)x := \{ x^* + y^* : x^* \in Sx, y^* \in Tx \}$. 

• $S + T$ is known as the “Minkowski sum” of $S$ and $T$. 

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— SN spaces, $r_L$–density and maximal monotonicity—

**Composite sum theorem**

Let $S, T: E \rightrightarrows E^*$ be closed, monotone and $r_L$–dense. Then (a)$\implies$(b)$\implies$(c)$\implies$(d):

(a) $D(S) \cap \text{int } D(T) \neq \emptyset$.

(b) $\bigcup_{\lambda > 0} \lambda \left[ D(S) - D(T) \right] = E$.

(c) $\bigcup_{\lambda > 0} \lambda \left[ \pi_1 \text{dom } \varphi_S - \pi_1 \text{dom } \varphi_T \right]$ is a closed subspace of $E$.

(d) $S + T$ is closed, monotone and $r_L$–dense.

**Composite parallel sum theorem**

Let $S, T: E \rightrightarrows E^*$ be closed, monotone and $r_L$–dense. Then (a)$\implies$(b)$\implies$(c)$\implies$(d):

(a) $R(S) \cap \text{int } R(T) \neq \emptyset$.

(b) $\bigcup_{\lambda > 0} \lambda \left[ R(S) - R(T) \right] = E^*$.

(c) $\bigcup_{\lambda > 0} \lambda \left[ \pi_2 \text{dom } \varphi_S - \pi_2 \text{dom } \varphi_T \right]$ is a closed subspace of $E^*$.

(d) The multifunction $y \mapsto (S^F + T^F)^{-1}(\hat{y})$ is closed, monotone and $r_L$–dense.

**Comments.** The two results above follow from the bivariate version of Fenchel duality. They are not immediate. These results are in stark contrast to the situation for maximally monotone multifunctions. Is is apparently still unknown whether $S + T$ is maximally monotone when $S$ and $T$ are maximally monotone and $D(S) \cap \text{int } D(T) \neq \emptyset$.  

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Composite sum theorem

Let $S, T: E \rightrightarrows E^*$ be closed, monotone and $r_L$–dense. Then (a)$\implies$(b)$\implies$(c)$\implies$(d):
(a) $D(S) \cap \text{int} \ D(T) \neq \emptyset$.
(b) $\bigcup_{\lambda > 0} \lambda \left[D(S) - D(T)\right] = E$.
(c) $\bigcup_{\lambda > 0} \lambda \left[\pi_1 \text{dom} \varphi_S - \pi_1 \text{dom} \varphi_T\right]$ is a closed subspace of $E$.
(d) $S + T$ is closed, monotone and $r_L$–dense.

Composite parallel sum theorem

Let $S, T: E \rightrightarrows E^*$ be closed, monotone and $r_L$–dense. Then (a)$\implies$(b)$\implies$(c)$\implies$(d):
(a) $R(S) \cap \text{int} \ R(T) \neq \emptyset$.
(b) $\bigcup_{\lambda > 0} \lambda \left[R(S) - R(T)\right] = E^*$.
(c) $\bigcup_{\lambda > 0} \lambda \left[\pi_2 \text{dom} \varphi_S - \pi_2 \text{dom} \varphi_T\right]$ is a closed subspace of $E^*$.
(d) The multifunction $y \mapsto (S^F + T^F)^{-1}(\hat{y})$ is closed, monotone and $r_L$–dense.

Another comment. If $S: E \rightrightarrows E^*$ is closed, monotone and $r_L$–dense then it can be proved that $S^F: E^* \rightrightarrows E^{**}$ is maximally monotone. This does not seem to be very easy. Our proof depends on the following result of Simon Fitzpatrick and myself:

On the biconjugate of a maximum

Let $X$ be a nonzero Banach space, $m \geq 1$, $g_0 \in \mathcal{PCLSC}(X)$ and $g_1, \ldots, g_m$ be convex and continuous on $X$. Then
\[
(g_0 \vee \cdots \vee g_m)^{**} = g_0^{**} \vee \cdots \vee g_m^{**}.
\]

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**First subdifferential perturbation theorem**

Let $S : E \rightrightarrows E^*$ be closed, monotone and $r_L$–dense. Let $k \in \mathcal{PCLSC}(E)$ and either $D(S) \cap \text{int dom } k \neq \emptyset$ or $\text{int } D(S) \cap \text{dom } k \neq \emptyset$. Then the multifunction $S + \partial k$ is closed, monotone and $r_L$–dense.

**Second subdifferential perturbation theorem**

Let $S : E \rightrightarrows E^*$ be closed, monotone and $r_L$–dense. Let $k \in \mathcal{PCLSC}(E)$ and either $R(S) \cap \text{int dom } k^* \neq \emptyset$ or $\text{int } R(S) \cap \text{dom } k^* \neq \emptyset$. Then the multifunction

$$y \mapsto (S^{\text{Inf}} + \partial k^*)^{-1}(\hat{y})$$

is closed, monotone and $r_L$–dense.

**Comments.** The first subdifferential perturbation theorem follows easily from the composite sum theorem. The second subdifferential perturbation theorem follows with a little more difficulty from the composite parallel sum theorem.
— SN spaces, $r_L$–density and maximal monotonicity—

**Strong maximality**

Let $S: E \rightrightarrows E^*$ be monotone. We say that $S$ is *strongly maximally monotone* if:

(a) Whenever $C$ is a nonempty $w(E^*,E)$–compact convex subset of $E^*$, $w \in E$ and,
\[ \forall (s,s^*) \in G(S), \ \exists w^* \in C \text{ such that } \langle s - w, s^* - w^* \rangle \geq 0 \]
then $S(w) \cap C \neq \emptyset$.

(b) Whenever $C$ is a nonempty $w(E,E^*)$–compact convex subset of $E$, $w^* \in E^*$ and,
\[ \forall (s,s^*) \in G(S), \ \exists w \in C \text{ such that } \langle s - w, s^* - w^* \rangle \geq 0 \]
then $w^* \in S(C)$.

**Strong maximality theorem**

Let $S: E \rightrightarrows E^*$ be closed, monotone and $r_L$–dense. Then $S$ is *strongly maximally monotone*.

**Comments.** Of course, if $C$ is a singleton, these statements become exactly the statement of maximal monotonicity.

(a) follows from the First subdifferential perturbation theorem with $k$ a support functional, and (b) follows from the Second subdifferential perturbation theorem with $k$ an indicator function.

This was another property originally proved for subdifferentials.

We do not have an example of a *maximally monotone* multifunction that is not strongly maximal.
— SN spaces, $r_L$–density and maximal monotonicity—

**Type (FP)**
Let $S: E \rightrightarrows E^*$ be monotone. We say that $S$ is of type (FP) provided that the following holds: if $U$ is an open convex subset of $E^*$, $U \cap R(S) \neq \emptyset$, $(w, w^*) \in E \times U$ and

$$\langle s - w, s^* - w^* \rangle \geq 0$$

whenever $(s, s^*) \in A$ and $s^* \in U$

then $(w, w^*) \in G(S)$.

• If we take $U = E^*$, we can see that every monotone multifunction of type (FP) is maximally monotone.
• This concept was originally introduced by Fitzpatrick and Phelps. Their term for it was “locally maximal monotone”.

**Type (FP) criterion for $r_L$–density**
Let $S: E \rightrightarrows E^*$ be closed and monotone. Then $S$ is of type (FP) $\iff S$ is $r_L$–dense.

**Comments.** “$\iff$” follows from the Second subdifferential perturbation theorem with $k$ a support functional.
“$\Rightarrow$” follows from an adaptation of a proof of Bauschke, Borwein, Wang and Yao.
— SN spaces, $r_L$–density and maximal monotonicity—

Polar subspace

- If $Y$ is a linear subspace of a Banach space $X$, $Y^0$ is the “polar subspace of $Y$”, that is to say the linear subspace $\{x^* \in X^* : \langle Y, x^* \rangle = \{0\}\}$ of $X^*$.

**Theorem on the $r_L$-density of linear subspaces**

Let $(B, L)$ be an SN space and $A$ be a closed linear $L$–positive subspace of $B$. Then $A$ is $r_L$-dense in $B$ $\iff$ $\sup s_L(A^0) \leq 0$.

- Let $A$ be a linear subspace of $E \times E^*$ (that is to say a linear relation). The adjoint subspace (adjoint linear relation), $A^T$, of $E^{**} \times E^*$, is defined by:

$$ (y^{**}, y^*) \in A^T \iff \text{ for all } (a, a^*) \in A, \langle a, y^* \rangle = \langle a^*, y^{**} \rangle. $$

(We use the notation “$A^T$” rather than the more usual “$A^*$” to avoid confusion with the dual space of $A$.) It is clear that

$$ (y^{**}, y^*) \in A^T \iff (y^*, -y^{**}) \in A^0. $$

Our next result extends the Brezis-Browder theorem to nonreflexive spaces.

**Generalization of results of Bauschke, Borwein, Wang and Yao**

Let $A$ be a closed linear monotone subspace of $E \times E^*$. Then the three conditions below are equivalent:

- $A$ is $r_L$–dense.
- $A^T$ is a monotone subspace of $E^{**} \times E^*$.
- $A^T$ is a maximally monotone subspace of $E^{**} \times E^*$.
— **SN spaces, $r_L$–density and maximal monotonicity**—

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<td>$\forall (x^<em>, x^{**}) \in E^</em> \times E^{<strong>}$, $\inf_{(s, s^<em>) \in G(S)} \langle s^</em> - x^*, \hat{s} - x^{</strong>}\rangle \leq 0$.</td>
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<th><strong>Type (NI) criterion for $r_L$–density</strong></th>
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<td>Let $S: E \rightrightarrows E^*$ be closed and monotone. Then $S$ is $r_L$–dense $\iff$ $S$ is maximally monotone of type (NI).</td>
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— SN spaces, \( r_L \)-density and maximal monotonicity—

Downloads

You can download files containing related materials and complete references from <www.math.ucsb.edu/~simons/SNRL.html>.

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