Discretization and Dualization of Linear-Quadratic Control Problems with Bang-Bang Solutions

Christopher Schneider

February 12, 2015
Where is Jena?
Is it a beautiful place?
Outline

Motivation: Optimal Control with ODEs

Linear-Quadratic Control Problems

Stability of Solutions

Discretization

The Dual Problem
Motivation: Optimal Control with ODEs

The Rocket Car
Motivation: Optimal Control with ODEs

A car powered by a rocket engine has to reach its aim as precise as possible in a given time $t_f > 0$. Therefore let $x_1(t)$ be the position, $x_2(t)$ the velocity and $u(t)$ the acceleration (control) of the car at time $t \in [0, t_f]$. 
Motivation: Optimal Control with ODEs

\[
\begin{align*}
\min & \quad \frac{1}{2} (x_1(t_f)^2 + x_2(t_f)^2) + \frac{\alpha}{2} \|u\|^2 \\
\text{s.t.} & \quad \dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = u(t) \quad \text{a.e. on } [0, t_f], \\
& \quad x_1(0) = 6, \quad x_2(0) = 1, \\
& \quad u(t) \in [-1, 1] \quad \text{a.e. on } [0, t_f].
\end{align*}
\]

\[x_1 \ldots \text{position of the car}\]
\[x_2 \ldots \text{velocity of the car}\]
\[u \ldots \text{acceleration of the car (control)}\]
Motivation: Optimal Control with ODEs

Optimal control for $\alpha = 0$ and $\alpha = 1$ ($t_f = 5$):

\[ u^0(t) = \begin{cases} 
-1, & \text{for } 0 \leq t \leq \tau, \\
1, & \text{for } \tau < t \leq t_f
\end{cases} \]

\[ u^\alpha(t) = \text{Pr}_{[-1,1]} \left[ -\frac{1}{\alpha} \lambda_2(t) \right] \]
Motivation: Optimal Control with ODEs

Parts 3 and 4 are joint work with Walter Alt and Martin Seydenschwanz.

Part 5 is joint work with Walter Alt and Yalcin Kaya.
Linear-Quadratic Control Problems

Basic Results
**Problem (PQ)**

\[
\min \ f(x, u)
\]

s.t. \[\dot{x}(t) = A(t)x(t) + B(t)u(t) + b(t) \quad \text{a.e. on } [t_0, t_f],\]
\[x(0) = x_0,\]
\[u(t) \in U := \{u \in \mathbb{R}^m \mid b_\ell \leq u \leq b_u\} \quad \text{a.e. on } [t_0, t_f],\]

where \( f \) is a linear-quadratic cost functional defined by

\[
f(x, u) = \frac{1}{2}x(t_f)^T Q x(t_f) + q^T x(t_f) + \int_{t_0}^{t_f} \frac{1}{2}x(t)^T W(t)x(t) + w(t)^T x(t) + x(t)^T S(t)u(t) + r(t)^T u(t) \, dt.
\]
Linear-Quadratic Control Problems

\( u(t) \in L^\infty(t_0, t_f; \mathbb{R}^m) \) is the control and \( x(t) \in W^1_\infty(t_0, t_f; \mathbb{R}^n) \) is the state of the system at time \( t \). The functions \( W, S, w, r, A, B, \) and \( b \) are Lipschitz continuous.

(AC) The matrices \( Q \) and \( W(t), t \in [t_0, t_f] \), are symmetric and

\[
 x(t_f)^T Q x(t_f) + \int_{t_0}^{t_f} x(t)^T W(t) x(t) + 2x(t)^T S(t) u(t) \, dt \geq 0
\]

for all \((x, u) \in X\) with

\[
 \dot{x}(t) = A(t)x(t) + B(t)u(t) \quad \text{a.e. on } [t_0, t_f], \\
x(t_0) = 0, \\
u(t) \in U - U \quad \text{a.e. on } [t_0, t_f].
\]

Then \((PQ)\) is a convex optimization problem and a solution exists.
Optimality Conditions

\((x^*, u^*)\) is a \textbf{solution} for Problem (PQ) iff there exists a function \(\lambda^*\) such that the \textbf{adjoint equation}

\[
-\dot{\lambda}^*(t) = A(t)^T \lambda^*(t) + W(t)x^*(t) + S(t)u^*(t) + w(t) \quad \text{a.e. on } [t_0, t_f],
\]

\[
\lambda^*(t_f) = Qx^*(t_f) + q,
\]

holds, and the \textbf{minimum principle}

\[
\left[ B(t)^T \lambda^*(t) + S(t)^T x^*(t) + r(t) \right]^T (u - u^*(t)) \geq 0 \quad \text{for all } u \in U
\]

is satisfied for a.e. \(t \in [t_0, t_f]\).
Linear-Quadratic Control Problems

Bang-Bang Structure

We denote the switching function by

$$\sigma(t) := B(t)^T \lambda^*(t) + S(t)^T x^*(t) + r(t).$$

Then the minimum principle implies for $i \in \{1, \ldots, m\}$

$$u^*_i(t) = \begin{cases} b_{\ell,i}, & \text{if } \sigma_i(t) > 0, \\ b_{u,i}, & \text{if } \sigma_i(t) < 0, \\ \text{undetermined}, & \text{if } \sigma_i(t) = 0. \end{cases}$$

If the switching function $\sigma$ has only finitely many isolated zeros, the optimal control $u^*$ is of bang-bang type.
(B1) The set $\Sigma$ of zeros of the components $\sigma_i, i = 1\ldots,m,$ of the switching function $\sigma$ is finite and $t_0, t_f \notin \Sigma,$ i.e., $\Sigma = \{s_1, \ldots, s_l\}$ with $t_0 < s_1 < \ldots < s_l < t_f$.

Let $\mathcal{I}(s_j) := \{1 \leq i \leq m \mid \sigma_i(s_j) = 0\}$ be the set of active indices for the components of the function $\sigma$. In order to obtain stability of the bang-bang structure under perturbations, we need an additional assumption:

(B2) There exist $\bar{\sigma} > 0, \bar{\tau} > 0$ such that

$$|\sigma_i(\tau)| \geq \bar{\sigma} |\tau - s_j|$$

for all $j \in \{1, \ldots, l\}, i \in \mathcal{I}(s_j),$ and all $\tau \in [s_j - \bar{\tau}, s_j + \bar{\tau}]$.

(B1) + (B2) $\Rightarrow$ Problem (PQ) has a unique solution of bang-bang type.
Part 3

Stability of Solutions

w.r.t. Perturbations and $L^2$-Regularization
Stability of Solutions

We introduce standard perturbations \( p = (\phi, \xi, \zeta, \eta) \), where

\[
\phi \in \mathbb{R}^n, \quad \xi \in L^\infty(t_0, t_f; \mathbb{R}^n), \quad \zeta \in L^\infty(t_0, t_f; \mathbb{R}^m), \quad \eta \in L^\infty(t_0, t_f; \mathbb{R}^n),
\]

and a regularization parameter \( \alpha \geq 0 \) and consider the \( L^2 \)-regularized parametric LQP

\[
\text{Problem } (PQ)^\alpha_p
\]

\[
\begin{align*}
\min & \quad f_p(x, u) + \frac{\alpha}{2} \| u \|_2^2 \\
\text{s.t.} & \quad \dot{x}(t) = A(t)x(t) + B(t)u(t) + b(t) + \eta(t) \quad \text{a.e. on } [t_0, t_f], \\
& \quad x(0) = x_0, \\
& \quad u(t) \in U := \{ u \in \mathbb{R}^m \mid b_\ell \leq u \leq b_u \} \quad \text{a.e. on } [t_0, t_f].
\end{align*}
\]
Here, $f_p$ is a linear-quadratic cost functional defined by

\[
f_p(x, u) = \frac{1}{2} x(t_f)^T Q x(t_f) + [q + \phi]^T x(t_f) \\
+ \int_{t_0}^{t_f} \frac{1}{2} x(t)^T W(t)x(t) + [w(t) + \xi(t)]^T x(t) \, dt \\
+ \int_{t_0}^{t_f} x(t)^T S(t)u(t) + [r(t) + \zeta(t)]^T u(t) \, dt.
\]

The parameters $p = 0$, $\alpha = 0$ are the reference parameters, and Problem (PQ) is the reference problem. We are interested in the behavior of solutions $(x_p^\alpha, u_p^\alpha)$ of Problem $(PQ)_p^\alpha$ in dependence of the parameters $p$ and $\alpha$. 
Stability of Solutions

Optimality Conditions

$(x_p^{\alpha}, u_p^{\alpha})$ is a solution for Problem $(PQ)^{\alpha}_h$ iff there exists a function $\lambda_p^{\alpha}$ such that the adjoint equation

$$-\dot{\lambda}_p^{\alpha}(t) = A(t)^T \lambda_p^{\alpha}(t) + W(t)x_p^{\alpha}(t) + S(t)u_p^{\alpha}(t) + w(t) + \xi(t) \quad \text{a.e.,}$$
$$\lambda_p^{\alpha}(t_f) = Qx_p^{\alpha}(t_f) + q + \phi,$$

holds, and the minimum principle

$$\left[\alpha u_p^{\alpha}(t) + B(t)^T \lambda_p^{\alpha}(t) + S(t)^T x_p^{\alpha}(t) + r(t) + \xi(t)\right]^T (u - u_p^{\alpha}(t)) \geq 0$$

is satisfied for all $u \in U$ and a.e. $t \in [t_0, t_f]$. 
Theorem (Calmness of Solutions)

Let Assumptions (AC), (B1) and (B2) be satisfied. Then there exist constants $c_u$, $c_x$ and $c_\lambda$ independent of $p$ and $\alpha$ such that for the optimal solutions of Problems $(PQ)_p^\alpha$ the estimates

$$
\|u_p^\alpha - u^*\|_1 \leq c_u (\|p\| + \alpha), \quad \|x_p^\alpha - x^*\|_{1,1} \leq c_x (\|p\| + \alpha)
$$

and

$$
\|\lambda_p^\alpha - \lambda^*\|_{1,1} \leq c_\lambda (\|p\| + \alpha)
$$

hold for all $p$ and $\alpha$. 

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\textit{L}^2\text{-Regularization}

We consider Problem $\text{(PQ)}_0^\alpha$ with the cost functional

$$f_0(x, u) + \frac{\alpha}{2} \|u\|_2^2.$$

This problem has a \textbf{unique Lipschitz continuous} optimal control:

$$u^\alpha(t) = \text{Pr}_{[b_L, b_u]} \left( -\frac{1}{\alpha} \left[ B(t)^T \lambda_0^\alpha(t) + S(t)^T x_0^\alpha(t) + r(t) \right] \right).$$

\textbf{Theorem}

For $\alpha \to 0$ we obtain $u_0^\alpha \to u^*$ and $x_0^\alpha \to x^*$. 

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Stability of Solutions

Optimal solution for $\alpha = 0$:
Stability of Solutions

Optimal solution for $\alpha = 1$:
Optimal solution for $\alpha = 0.5$:
Stability of Solutions

Optimal solution for $\alpha = 0.1$:
Stability of Solutions

For two functions \( u_1, u_2 \in L^\infty(t_0, t_f; \mathbb{R}^m) \) we define

\[
d^\#(u_1, u_2) := \text{meas}\{ t \in [t_0, t_f] \mid u_1(t) \neq u_2(t) \}.
\]

Theorem

Let Assumptions (AC), (B1) and (B2) be satisfied. Then there exists a constant \( \kappa \) independent of \( (p, \alpha) \) such that for any solution \( (x_p^\alpha, u_p^\alpha) \) of Problems \( (PQ)_p^\alpha \) the estimate

\[
d^\#(u_p^\alpha, u^*) \leq \kappa (\|p\| + \alpha)
\]

holds, if \( \|p\| + \alpha \) is sufficiently small.
Discretization and Dualization of LQPs

Part 4

Discretization

Explicit and Implicit Euler Method
Discretization

The discretization of an optimal control problem depends on the choice of the discretization scheme for the system equation. This section is devoted to the **Euler discretization** of Problem \((PQ)_{\alpha}^0\).

This results in the following finite dimensional optimization problem:

**Problem \((PQ)_{\alpha}^h\)**

\[
\begin{align*}
\min & \quad f_h(x_h, u_h) \\
\text{s.t.} & \quad x_{h,j+1} = x_{h,j} + h \left[ A(t_j)x_{h,j} + B(t_j)u_{h,j} + b(t_j) \right], \quad j \in \mathcal{J}_0^{N-1} \\
& \quad x_{h,0} = x_0, \\
& \quad u_{h,j} \in U, \quad j \in \mathcal{J}_0^{N-1}
\end{align*}
\]
Discretization

\[ f_h^\alpha(x_h, u_h) = \frac{1}{2} x_{h,N}^T Q x_{h,N} + q^T x_{h,N} \]

\[ + h \sum_{j=0}^{N-1} \left[ \frac{1}{2} x_{h,j}^T W(t_j) x_{h,j} + w(t_j)^T x_{h,j} + x_{h,j}^T S(t_j) u_{h,j} \right] \]

\[ + h \sum_{j=0}^{N-1} \left[ r(t_j)^T u_{h,j} + \frac{\alpha}{2} u_{h,j}^T u_{h,j} \right] \]

A solution \((x_h^\alpha, u_h^\alpha)\) of Problem \(\text{(PQ)}_h^\alpha\) exists. Since it may happen that one of the zeros of the discrete switching function is a discretization point, the optimal control has not to be unique in the case \(\alpha = 0\). For \(\alpha > 0\), the optimal control is uniquely determined.
We are able to show that $(x_h^\alpha, u_h^\alpha)$ also solves some Problem $(PQ)^{\alpha}_{p_h}$ for a parameter $p_h^\alpha = (\phi_h^\alpha, \xi_h^\alpha, \eta_h^\alpha)$ with

$$\| p_h^\alpha \| \leq c h,$$

where the constant $c$ is independent of $N \in \mathbb{N}$ and $\alpha \geq 0$.

Therefore, we apply the calmness result to prove convergence of the discretization.

In the same way, convergence of the implicit Euler discretization can be proved.
Discretization

Theorem (Convergence)
Let Assumptions (AC), (B1) and (B2) be satisfied. For any $N \in \mathbb{N}$, the corresponding mesh size $h = (t_f - t_0)/N$ and any $\alpha \geq 0$, and for any solution $(x_h^\alpha, u_h^\alpha)$ with associated multiplier $\lambda_h^\alpha$ the error estimate

$$\|u_h^\alpha - u^*\|_1 + \|x_h^\alpha - x^*\|_{1,1} + \|\lambda_h^\alpha - \lambda^*\|_{1,1} \leq c (h + \alpha)$$

holds with some constant $c$ independent of $N$.

With some constant $\gamma$ we choose $\alpha = \gamma h$ and obtain

$$\|u_h^\alpha - u^*\|_1 + \|x_h^\alpha - x^*\|_{1,1} + \|\lambda_h^\alpha - \lambda^*\|_{1,1} \leq \tilde{c} h.$$
Theorem
Let Assumptions (AC), (B1) and (B2) be satisfied. There exists a constant $\kappa$ independent of $N \in \mathbb{N}$, the corresponding mesh size $h$ and $\alpha \geq 0$ such that

$$d^\#(u^\alpha_h, u^*) \leq \kappa (h + \alpha)$$

holds for all $N \in \mathbb{N}$ and $\alpha \geq 0$. 
Numerical Results for the Rocket Car Example

**Explicit Euler vs. Implicit Euler, \( N = 25 \):**

![Graphs showing numerical results for Explicit Euler vs. Implicit Euler.](image-url)
Numerical Results for the Rocket Car Example

Explicit Euler vs. Implicit Euler, $N = 50$:
Numerical Results for the Rocket Car Example

Explicit Euler vs. Implicit Euler, $N = 100$:

![Graphs showing the comparison between Explicit Euler and Implicit Euler for the Rocket Car Example.](image)
Numerical experiments confirm the theoretical findings. Here, $u_{\text{Exp}}$ is the solution of the (explicit) Euler discretized problem and $u_{\text{Imp}}$ is the solution of the implicit Euler discretized problem.

<table>
<thead>
<tr>
<th>N</th>
<th>100</th>
<th>200</th>
<th>400</th>
<th>800</th>
<th>1600</th>
<th>3200</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|u_{\text{Exp}} - u^*|_1$</td>
<td>0.1167</td>
<td>0.0602</td>
<td>0.0293</td>
<td>0.0151</td>
<td>0.0074</td>
<td>0.0036</td>
</tr>
<tr>
<td>$|u_{\text{Ex}}p - u^*|_1 / h$</td>
<td>2.3343</td>
<td>2.4085</td>
<td>2.3408</td>
<td>2.4227</td>
<td>2.3798</td>
<td>2.2873</td>
</tr>
<tr>
<td>$|u_{\text{Imp}} - u^*|_1$</td>
<td>0.0805</td>
<td>0.0421</td>
<td>0.0202</td>
<td>0.0106</td>
<td>0.0052</td>
<td>0.0026</td>
</tr>
<tr>
<td>$|u_{\text{Imp}} - u^*|_1 / h$</td>
<td>1.6105</td>
<td>1.6826</td>
<td>1.6160</td>
<td>1.7007</td>
<td>1.6547</td>
<td>1.6906</td>
</tr>
</tbody>
</table>
Discretization

A Stiff Optimal Control Problem

\[
\begin{align*}
\min & \quad \int_0^1 2x_1(t) + 6x_2(t) - u_1(t) - 0.5u_2(t) \, dt \\
\text{s. t.} & \quad \dot{x}_1(t) = 0.5 (c_1 + c_2) x_1(t) + 0.5 (c_1 - c_2) x_2(t) + u_1(t) \quad \text{a.e.,} \\
& \quad \dot{x}_2(t) = 0.5 (c_1 - c_2) x_1(t) + 0.5 (c_1 + c_2) x_2(t) + u_2(t) \quad \text{a.e.,} \\
& \quad x_1(0) = 0, \quad x_2(0) = 0, \\
& \quad u_1(t) \in [-1, 1], \quad u_2(t) \in [-2, 2] \quad \text{a.e.}
\end{align*}
\]

We choose \( c_1 = -1 \) and \( c_2 = -1000 \), so the problem becomes stiff.
Discretization

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Part 5

The Dual Problem

Continuous-Time Problem and Discretization
The Dual Problem

Burachik/Kaya/Majeed (SICON, 2014)

Computation of the dual problem for linear-quadratic control problems with continuous solutions. Strong duality holds.

Numerical experiments illustrate that by solving the dual problem computational savings can be achieved.
The Dual Problem

\[
\begin{align*}
\min & \quad \frac{1}{2} \left( p(t_f) + q \right)^T Q^{-1} \left( p(t_f) + q \right) + p(0)^T a \\
& \quad + \int_{t_0}^{t_f} \frac{1}{2} \left( x^*(t) - w(t) \right)^T W(t)^{-1} \left( x^*(t) - w(t) \right) + \psi^\alpha(p(t), t) \, dt \\
\text{s.t.} & \quad \dot{p}(t) = -A(t)^T p(t) + x^*(t) \quad \text{a.e. on } [t_0, t_f],
\end{align*}
\]

where for the \( \alpha = 0 \) (bang-bang case) \( \psi^0(p, t) \) is defined by

\[
\psi^0_i(p, t) = \begin{cases} 
  b_{\ell,i} \left( B(t)^T p \right)_i - b_{\ell,i} r_i(t) & \text{if } r_i(t) - \left( B(t)^T p \right)_i \geq 0, \\
  b_{u,i} \left( B(t)^T p \right)_i - b_{u,i} r_i(t) & \text{if } r_i(t) - \left( B(t)^T p \right)_i < 0.
\end{cases}
\]
The Dual Problem

The Regularized Case

Remember ($\alpha = 0$)

$$\psi_i^0(p, t) = \begin{cases} 
  b_{\ell,i} \left( (B(t)^T p) \right)_i - b_{\ell,i} r_i(t), & \text{if } r_i(t) - (B(t)^T p)_i \geq 0, \\
  b_{u,i} \left( (B(t)^T p) \right)_i - b_{u,i} r_i(t), & \text{if } r_i(t) - (B(t)^T p)_i < 0.
\end{cases}$$

For the $\alpha > 0$ (regularized case) $\psi_\alpha^\alpha(p, t)$ is defined by

$$\psi_i^\alpha(p, t) = \begin{cases} 
  \frac{1}{2\alpha} \left( (B(t)^T p) \right)_i^2 - \frac{1}{\alpha} r_i(t) \left( (B(t)^T p) \right)_i + \frac{1}{2\alpha} r_i(t)^2, & \text{if } r_i(t) \left( (B(t)^T p) \right)_i \\
  b_{\ell,i} \left( (B(t)^T p) \right)_i - b_{\ell,i} r_i(t) - \frac{\alpha}{2} b_{\ell,i}^2, & \text{if } r_i(t) \left( (B(t)^T p) \right)_i < 0, \\
  b_{u,i} \left( (B(t)^T p) \right)_i - b_{u,i} r_i(t) - \frac{\alpha}{2} b_{u,i}^2, & \text{if } r_i(t) - (B(t)^T p)_i < 0.
\end{cases}$$
**Theorem (Strong Duality)**

Let \((x, u)\) be an optimal solution of the primal problem with adjoint variable \(\lambda\). We set

\[
p(t) = -\lambda(t) \quad \text{a.e. on } [t_0, t_f], \quad p(t_f) = -Qx(t_f) - q,
\]

and

\[
x^*(t) = W(t)x(t) + w(t) \quad \text{a.e. on } [t_0, t_f].
\]

Then the optimal values of the primal and dual problem are equal and \((p, x^*)\) is a solution of the dual problem.
The Dual Problem

Discretization of the Dual Problem

\[ \|u_h^* - u^*\|_1 \leq ch \]

\[ (PQ) \quad \leftrightarrow \quad \text{Strong Duality} \quad \leftrightarrow \quad (DP) \]

\[ (PQ)_h \quad \leftrightarrow \quad \text{Strong Duality} \quad \leftrightarrow \quad (DP)_h \]
Revisiting the Rocket Car

\[ \min \frac{1}{2} (x_1(t_f)^2 + x_2(t_f)^2) + \frac{\alpha}{2} \|u\|_2^2 \]

s.t. \[ \dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = u(t) \quad \text{a.e. on } [0, t_f], \]
\[ x_1(0) = 6, \quad x_2(0) = 1, \]
\[ u(t) \in [-1, 1] \quad \text{a.e. on } [0, t_f]. \]
The Dual Problem

<table>
<thead>
<tr>
<th>Problem</th>
<th>Regularization</th>
<th>CPU time [s]</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primal</td>
<td>$\alpha = 0$</td>
<td>0.264</td>
<td>100%</td>
</tr>
<tr>
<td>Primal</td>
<td>$\alpha = h$</td>
<td>0.283</td>
<td>107%</td>
</tr>
<tr>
<td>Dual</td>
<td>$\alpha = 0$</td>
<td>0.192</td>
<td>73%</td>
</tr>
<tr>
<td>Dual</td>
<td>$\alpha = h$</td>
<td>0.150</td>
<td>57%</td>
</tr>
</tbody>
</table>

$N = 5000$, average over 1000 runs. Solver: IPOPT.
The Dual Problem

Diabetes Mellitus

\[
\begin{align*}
\min \quad & \int_0^1 \frac{1}{2} x_1(t)^2 \, dt + \frac{\alpha}{2} \|u\|_2^2 \\
\text{s.t.} \quad & \dot{x}_1(t) = -0.1x_1(t) - x_2(t) \quad \text{a.e. on } [0, 1], \\
& \dot{x}_2(t) = 0.2x_1(t) + 0.1x_2(t) + u(t) \quad \text{a.e. on } [0, 1], \\
& x_1(0) = 1, \quad x_2(0) = 0, \\
& u(t) \in [0, 4] \quad \text{a.e. on } [0, 1]
\end{align*}
\]
The Dual Problem

<table>
<thead>
<tr>
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<th>Regularization</th>
<th>CPU time [s]</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primal</td>
<td>$\alpha = 0$</td>
<td>0.576</td>
<td>100%</td>
</tr>
<tr>
<td>Primal</td>
<td>$\alpha = h$</td>
<td>0.359</td>
<td>61%</td>
</tr>
<tr>
<td>Dual</td>
<td>$\alpha = 0$</td>
<td>0.211</td>
<td>37%</td>
</tr>
<tr>
<td>Dual</td>
<td>$\alpha = h$</td>
<td>0.182</td>
<td>32%</td>
</tr>
</tbody>
</table>

$N = 5000$, average over 1000 runs. Solver: IPOPT.
References


Questions?

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