A new subfamily of enlargements of a maximally monotone operator

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Motivation

Preliminaries

The family $\mathcal{H}(T)$

Enlargements of $T$

Case $T = \partial \varphi$
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1 Motivation
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5 Case $T = \partial \varphi$
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Monotone Inclusion Problem

Let $T : X \rightrightarrows X^*$ be maximal monotone. Many nonlinear problems are stated as:

Given $z \in X^*$, find $x \in X : z \in T(x)$  \hspace{1cm} (P_0)

Equivalently:

Given $z \in X^*$, find $x \in X : (x, z) \in G(T)$

solving $(P_0) \iff$ requires to know $G(T)$
Main Ingredients I: multivalued mappings

For $T : X \nRightarrow X^*$ we define

- its graph as $G(T) := \{(x, x^*) \in X \times X^* : x^* \in T(x)\}$,
- its domain as $D(T) := \{x \in X : T(x) \neq \emptyset\}$,
- its range as $R(T) := \bigcup\{T(x) : x \in D(T)\}$,

We say that $T$ is

- *monotone* if
  \[
  \langle y - x, y^* - x^* \rangle \geq 0 \quad \forall (x, x^*), (y, y^*) \in G(T).
  \]

- *maximally monotone* if $T$ has no monotone extension in the sense of graph inclusion.
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Main Ingredients II: subdifferentials

For $\varphi : X \rightarrow \mathbb{R}_\infty$ convex and lsc, we define

- $\text{Dom} \varphi := \{ x : \varphi(x) < \infty \}$, and
- we say that $\varphi$ is proper when $\text{Dom} \varphi \neq \emptyset$.
- the subdifferential of $\varphi$ is the multivalued mapping $\partial \varphi : X \rightrightarrows X^*$ defined by

$$\partial \varphi(x) := \{ x^* \in X^* : \varphi(y) - \varphi(x) \geq \langle x^*, y - x \rangle, \forall y \in X \},$$

when $x \in \text{Dom} \varphi$. Otherwise $\partial \varphi(x) = \emptyset$. 
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Fenchel Young inequality

Let \( \varphi : X \to \mathbb{R}_\infty \) be convex and lsc, \( \varphi^* : X^* \to \mathbb{R}_\infty \)

\[
\varphi^*(v) := \sup_{x \in X} \{ \langle x, v \rangle - \varphi(x) \}
\]

is the \textit{conjugate of} \( \varphi \). The \textit{Fenchel Young inequality} states

\[
\varphi(x) + \varphi^*(v) \geq \langle x, v \rangle, \quad \forall \ x \in X, \ v \in X^*
\]

\[
\varphi(x) + \varphi^*(v) = \langle x, v \rangle, \quad \iff \quad v \in \partial \varphi(x).
\]

Notation:

\[
\varphi^{FY}(x, v) := \varphi(x) + \varphi^*(v)
\]
Fitzpatrick Theory: the family $\mathcal{H}(T)$

In 1988 Fitzpatrick defined the family $\mathcal{H}(T)$ consisting of all $h : X \times X^* \to \mathbb{R}_\infty$ convex and lsc such that:

\[
\begin{align*}
    h(x, v) &\geq \langle x, v \rangle, \quad \forall \ x \in X, \ v \in X^* \\
    h(x, v) &= \langle x, v \rangle, \iff \ v \in T(x).
\end{align*}
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Given $v$ this reformulates the monotone inclusion as an optimization problem in $X$: Find $x$ such that

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h(x, v) = 0 = \min_x h(\cdot, v)
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A key member of $\mathcal{H}(T)$

Fitzpatrick defined $\mathcal{F}_T : X \times X^* \rightarrow \mathbb{R}_\infty$ as

$$\mathcal{F}_T(x, x^*) := \sup_{(y, y^*) \in G(T)} \langle y, x^* \rangle + \langle x - y, y^* \rangle$$

which verifies

- $\mathcal{F}_T \in \mathcal{H}(T)$
- $\mathcal{F}_T \leq h \leq (\mathcal{F}_T)^* =: \sigma_T$ for all $h \in \mathcal{H}(T)$

Historical note: N.V.Krylov defined in 1980 $\mathcal{F}_T$ for $T$ point-to-point monotone in finite dimensions.
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For $\varphi : X \to \mathbb{R}_\infty$ convex, lsc, let $\varepsilon \geq 0$, then $\partial_\varepsilon \varphi : X \rightrightarrows X^*$ is

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\partial_\varepsilon \varphi (x) := \{ x^* \in X^* : \varphi(y) - \varphi(x) \geq \langle x^*, y-x \rangle - \varepsilon, \forall y \in X \},
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if $x \in \text{Dom} \varphi$. Otherwise, $\partial_\varepsilon \varphi (x) = \emptyset$.

$\tilde{\partial} \varphi (\varepsilon, x) := \partial_\varepsilon \varphi (x)$ Brøndsted-Rockafellar enlargement (1965)

$\tilde{\partial} \varphi$ characterized by Fenchel Young ineq.:

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The family $E(T)$ of enlargements of $T$

$E : \mathbb{R}_+ \times X \Rightarrow X^*$ is in $E(T)$ when

$(E_1)$ $T(x) \subset E(\varepsilon, x)$ for all $\varepsilon \geq 0, x \in X$;

$(E_2)$ If $0 \leq \varepsilon_1 \leq \varepsilon_2$, then $E(\varepsilon_1, x) \subset E(\varepsilon_2, x)$ for all $x \in X$;

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\bar{x} := \alpha_1 x^1 + \alpha_2 x^2, \quad \bar{v} := \alpha_1 v^1 + \alpha_2 v^2 \text{ and }
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\bar{\varepsilon} := \alpha_1 \varepsilon_1 + \alpha_2 \varepsilon_2 + \alpha_1 \alpha_2 \langle v^1 - v^2, x^1 - x^2 \rangle,
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then $\bar{\varepsilon} \geq 0$ and $\bar{v} \in E(\bar{\varepsilon}, \bar{x})$.

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Example: $\partial\varphi \in \mathbb{E}(\partial\varphi)$
From enlargements to convex functions:

\[(E_3) \iff \tilde{G}(E) \text{ convex,}\]

where

\[
G(E) := \{ (x, v, \varepsilon) : v \in E(\varepsilon, x) \}
\]

\[
\tilde{G}(E) := \{ (x, v, \varepsilon + \langle x, v \rangle) : v \in E(\varepsilon, x) \}
\]
From $\mathcal{E}(T)$ to $\mathcal{H}(T)$

$E \in Enl(T) \iff \tilde{G}(E)$ is the \begin{cases} \text{epigraph of a lsc. convex function} \\ \text{on } X \times X^* \end{cases}

This convex function is given by

$$h_E(x, v) := \inf \{ t : (x, v, t) \in \tilde{G}(E) \}$$

Moreover, $h_E \in \mathcal{H}(T)$ for all $E \in \mathcal{E}(T)$!
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$$L^h(\varepsilon, x) := \{v \in X^* : h(x, v) \leq \langle x, v \rangle + \varepsilon\}$$

Then $L^h \in \mathcal{E}(T)$ for all $h \in \mathcal{H}(T)$!

$\mathcal{H}(T) \leftrightarrow \mathcal{E}(T)$

Case $T = \partial \varphi$

Recall $\varphi^{FY}(x, v) = \varphi(x) + \varphi^*(v)$, then $\varphi^{FY} \in \mathcal{H}(\partial \varphi)$
**Extreme members in the families**

\( \mathcal{H}(T) \) has a smallest and a largest element

\[ \mathcal{F}_T \leq h \leq \sigma_T = (\mathcal{F}_T)^*, \quad \mathbb{E}(T) \text{ has largest element:} \]

\[ T^{BE}(\varepsilon, x) := \{ v \in X^* : \langle x - y, v - u \rangle \geq -\varepsilon, \forall (y, u) \in G(T) \}, \]

and smallest \( T^{SE}(\varepsilon, x) = \bigcap_{E \in \mathbb{E}(T)} E(\varepsilon, x), \)

Related through \( L^{\mathcal{F}_T} = T^{BE}, \) and \( L^{\sigma_T} = T^{SE} \)

\[ h_{T^{SE}} = \sigma_T, \text{ and } h_{T^{BE}} = \mathcal{F}_T \]

**Question:** Can we identify a property that singles out “nice” enlargements?
Extreme members in the families

\( \mathcal{H}(T) \) has a smallest and a largest element
\( \mathcal{F}_T \leq h \leq \sigma_T = (\mathcal{F}_T)^* \), \( \mathbb{E}(T) \) has largest element:

\[ T^{BE}(\varepsilon, x) := \{ v \in X^* : \langle x - y, v - u \rangle \geq -\varepsilon, \forall (y, u) \in G(T) \}, \]

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\[ h_{T^{SE}} = \sigma_T \], and \( h_{T^{BE}} = \mathcal{F}_T \)

Question: Can we identify a property that singles out “nice” enlargements?
Extreme members in the families

\[ \mathcal{H}(T) \] has a smallest and a largest element
\[ F_T \leq h \leq \sigma_T = (F_T)^*, \quad \mathbb{E}(T) \] has largest element:

\[ T^{BE}(\varepsilon, x) := \{ v \in X^* : \langle x - y, v - u \rangle \geq -\varepsilon, \forall (y, u) \in \mathcal{G}(T) \}, \]

and smallest \[ T^{SE}(\varepsilon, x) = \bigcap_{E \in \mathbb{E}(T)} E(\varepsilon, x), \]

Related through \[ L^{F_T} = T^{BE}, \quad \text{and} \quad L^{\sigma_T} = T^{SE} \]

\[ h_{T^{SE}} = \sigma_T, \quad \text{and} \quad h_{T^{BE}} = F_T \]

Question: Can we identify a property that singles out “nice” enlargements?
Extreme members in the families

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\[ h_{T^{SE}} = \sigma_T, \quad \text{and} \quad h_{T^{BE}} = \mathcal{F}_T \]

Question: Can we identify a property that singles out “nice” enlargements?
Additivity

- \( E \in \mathbb{E}(T) \) is additive, if

\[
\langle v_1 - v_2, x_1 - x_2 \rangle \geq - (\varepsilon_1 + \varepsilon_2).
\]

Set \( \mathbb{E}_a(T) := \{ E \in \mathbb{E}(T) : E \text{ additive} \} \)

\( \partial \varphi \) is additive, i.e., \( \partial \varphi \in \mathbb{E}_a(\partial \varphi) \)

\( T^{SE} \) is always additive, but \( T^{BE} \) may not!

Additivity detects those elements in \( \mathbb{E}(T) \) which have something in common with \( \partial \varphi \)!
Additivity

- $E \in \mathbb{E}(T)$ is *additive*, if

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Additivity

$E \in \mathbb{E}(T)$ is additive, if

$$\nu_1 \in E(\varepsilon_1, x_1), \; \nu_2 \in E(\varepsilon_1, x_2)$$

$$\langle \nu_1 - \nu_2, x_1 - x_2 \rangle \geq - (\varepsilon_1 + \varepsilon_2).$$

Set $\mathbb{E}_a(T) := \{ E \in \mathbb{E}(T) : E \text{ additive} \}$

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Additivity

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Set \( \mathbb{E}_a(T) := \{ E \in \mathbb{E}(T) : E \text{ additive} \} \)

\( \partial \phi \) is additive, i.e., \( \partial \phi \in \mathbb{E}_a(\partial \phi) \)

\( T^{SE} \) is always additive, but \( T^{BE} \) may not!

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Additivtty

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Set $\mathbb{E}_a(T) := \{ E \in \mathbb{E}(T) : E \text{ additive} \}$

$\partial \varphi$ is \textit{additive}, i.e., $\partial \varphi \in \mathbb{E}_a(\partial \varphi)$

$T^{SE}$ is always \textit{additive}, but $T^{BE}$ may not!

Additivity detects those elements in $\mathbb{E}(T)$ which have something in common with $\partial \varphi$!
Additivity as a mutual relation/maximal property

- \( E \in \mathbb{E}_a(T) \) is **maximally additive** *(max-add, for short)*, if

  \[
  \exists \hat{E} \in \mathbb{E}_a(T) : E(\varepsilon, x) \subset \hat{E}(\varepsilon, x), \forall \varepsilon \geq 0, \forall x \in X
  \]

  \[
  \Downarrow
  
  E = \hat{E}
  \]

- \( E_1, E_2 \in \mathbb{E}(T) \) are **mutually additive**, if

  \[
  v_1 \in E_1(\varepsilon_1, x_1), v_2 \in E_2(\varepsilon_1, x_2)
  \]

  \[
  \Downarrow
  
  \langle v_1 - v_2, x_1 - x_2 \rangle \geq -(\varepsilon_1 + \varepsilon_2).
  \]

  Denoted as \( E_1 \sim_a E_2 \implies E \sim_a E \) iff \( E \in \mathbb{E}_a(T) \)
Additivity as a mutual relation/maximal property

- \( E \in \mathbb{E}_a(T) \) is **maximally additive** (\textit{max-add}, for short), if

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\]

- \( E_1, E_2 \in \mathbb{E}(T) \) are **mutually additive**, if

\[
\begin{align*}
\nu_1 & \in E_1(\varepsilon_1, x_1), \nu_2 \in E_2(\varepsilon_1, x_2) \\
\Downarrow \\
\langle \nu_1 - \nu_2, x_1 - x_2 \rangle & \geq -(\varepsilon_1 + \varepsilon_2).
\end{align*}
\]

Denoted as \( E_1 \sim_a E_2 \implies E \sim_a E \iff E \in \mathbb{E}_a(T) \)
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  \[
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  \[
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  \[
  \Downarrow \quad \langle v_1 - v_2, x_1 - x_2 \rangle \geq -(\varepsilon_1 + \varepsilon_2). \]

  Denoted as \( E_1 \sim_a E_2 \quad \Rightarrow \quad E \sim_a E \text{ iff } E \in \mathbb{E}_a(T) \)
Additively as a mutual relation/maximal property

- \( E \in \mathbb{E}_a(T) \) is \textit{maximally additive} (\textit{max-add}, for short), if
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  \[
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  \]
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Denoted as \( E_1 \sim_a E_2 \implies E \sim_a E \) iff \( E \in \mathbb{E}_a(T) \).
Additivity as a mutual relation/maximal property

- $E \in \mathbb{E}_a(T)$ is maximally additive (max-add, for short), if
  \[ \exists \hat{E} \in \mathbb{E}_a(T) : E(\varepsilon, x) \subset \hat{E}(\varepsilon, x), \forall \varepsilon \geq 0, \forall x \in X \]
  \[ \Downarrow \]
  \[ E = \hat{E} \]

- $E_1, E_2 \in \mathbb{E}(T)$ are mutually additive, if
  \[ v_1 \in E_1(\varepsilon_1, x_1), v_2 \in E_2(\varepsilon_1, x_2) \]
  \[ \Downarrow \]
  \[ \langle v_1 - v_2, x_1 - x_2 \rangle \geq - (\varepsilon_1 + \varepsilon_2). \]

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  \]

  Denoted as $E_1 \sim_a E_2 \implies E \sim_a E$ iff $E \in \mathbb{E}_a(T)$
Example of Max-Additivity

If $T = ∂φ$ then $\tilde{∂Φ}$ is max-add (Svaiter, 2000)

If $T$ arbitrary, then $T^{SE}$ is always additive, but not necessarily max-add!

Max-additivity detects those elements in $E_a(T)$ which have even more in common with $\tilde{∂φ}$!
Example of Max-Additivity

If $T = \partial \varphi$ then $\tilde{\partial} \varphi$ is max-add (Svaiter, 2000)

If $T$ arbitrary, then $T^{SE}$ is always additive, but not necessarily max-add!

Max-additivity detects those elements in $E_a(T)$ which have even more in common with $\tilde{\partial} \varphi$!
Motivation
Preliminaries
The family $\mathcal{H}(T)$
Enlargements of $T$
Case $T = \partial \varphi$

Additive enlargements
Mutual additivity
New enlargements

Example of Max-Additivity

If $T = \partial \varphi$ then $\tilde{\partial} \varphi$ is max-add (Svaiter, 2000)

If $T$ arbitrary, then $T^{SE}$ is always additive, but not necessarily max-add!

Max-additivity detects those elements in $E_a(T)$ which have even more in common with $\tilde{\partial} \varphi$!
Example of mutual additivity

If \( T \) arbitrary, then \( T^{SE} \) and \( T^{BE} \) are always mutually additive (Svaiter, 2000)

**Questions:** How to identify additive elements \( E(T) \)? How to identify max-add elements within \( E_a(T) \)? How to characterize mutual additivity?

We will address these using convex functions!
Let $f : X \times X^* \to \mathbb{R}_\infty$ be convex, Fitzpatrick (1988) defined $T_f : X \Rightarrow X^*$ as

$$T_f(x) := \{v \in X^* : (v, x) \in \partial f(x, v)\} \star$$

Fitzpatrick proved that $T_f$ mon, and for $T$ monotone and $f := \mathcal{F}_T$:

- $\forall x \in X, \ T(x) \subseteq T_{\mathcal{F}_T}(x)$.
- $T$ maximal $\implies T = T_{\mathcal{F}_T}$.

Can recover $T$ as a diagonal slice of the $\partial \mathcal{F}_T$!

**Question:** What happens if we use $\partial f$ in $\star$? Can we still recover $T$?
From convex functions to $T$ and viceversa

Let $f : X \times X^* \to \mathbb{R}_\infty$ be convex, Fitzpatrick (1988) defined $T_f : X \Rightarrow X^*$ as

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From convex functions to $T$ and viceversa

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- $T$ maximal $\implies T = T_{F_T}$

Can recover $T$ as a diagonal slice of the $\partial F_T$!

Question: What happens if we use $\tilde{\partial} f$ in $\star$?
Can we still recover $T$?
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Can recover $T$ as a diagonal slice of the $\partial \mathcal{F}_T$!

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- $T$ maximal $\implies T = T_{\mathcal{F}_T}$

Can recover $T$ as a diagonal slice of the $\partial \mathcal{F}_T$!

**Question:** What happens if we use $\tilde{\partial}f$ in $\star$? Can we still recover $T$?
Let $h \in \mathcal{H}(T)$, define $\mathcal{J} : \mathcal{H}(T) \rightarrow \mathcal{H}(T)$ as

$$\mathcal{J}h(x, v) := h^*(v, x)$$

I.e., $\mathcal{J}h$ swaps the variables of $h^*$

Define $A : \mathcal{H}(T) \rightarrow \mathcal{H}(T)$ as

$$Ah := \frac{h + \mathcal{J}h}{2}$$

Fact: $Ah \in \mathcal{H}(T)$ if $h \in \mathcal{H}(T)$. 
Let $h \in \mathcal{H}(T)$, define $\mathcal{J} : \mathcal{H}(T) \rightarrow \mathcal{H}(T)$ as

$$\mathcal{J}h(x, v) := h^*(v, x)$$

i.e., $\mathcal{J}h$ swaps the variables of $h^*$. Define $\mathcal{A} : \mathcal{H}(T) \rightarrow \mathcal{H}(T)$ as

$$\mathcal{A}h := \frac{h + \mathcal{J}h}{2}$$

Fact: $\mathcal{A}h \in \mathcal{H}(T)$ if $h \in \mathcal{H}(T)$. 
An induced subfamily of enlargements

Let $T$ be max-mon and fix $h \in \mathcal{H}(T)$. We define

\[ \tilde{T}_h : \mathbb{R}_+ \times X \Rightarrow X^* \] as

\[ \tilde{T}_h(\varepsilon, x) := \{ v \in X^* : (v, x) \in \partial h(2\varepsilon, x, v) \} \]

- $T_h(x) = \tilde{T}_h(0, x) = T$
- $\tilde{T}_h = L^A h$, so $\tilde{T}_h \in E(T)$.

Define $E_{\mathcal{H}}(T) := \{ E \in E(T) : E = \tilde{T}_h \text{ for some } h \in \mathcal{H}(T) \}$

Question: $E_{\mathcal{H}}(T)$ has special properties, not shared by other elements of $E(T)$?
An induced subfamily of enlargements

Let $T$ be max-mon and fix $h \in \mathcal{H}(T)$. We define

$$
\tilde{T}_h : \mathbb{R}_+ \times X \Rightarrow X^* \text{ as }
$$

$$
\tilde{T}_h(\varepsilon, x) := \{ v \in X^* : (v, x) \in \partial h(2\varepsilon, x, v) \}
$$

- $T_h(x) = \tilde{T}_h(0, x) = T$
- $\tilde{T}_h = L^{Ah}$, so $\tilde{T}_h \in \mathcal{E}(T)$.

Define $\mathcal{E}_{\mathcal{H}}(T) := \{ E \in \mathcal{E}(T) : E = \tilde{T}_h \text{ for some } h \in \mathcal{H}(T) \}$

Question: $\mathcal{E}_{\mathcal{H}}(T)$ has special properties, not shared by other elements of $\mathcal{E}(T)$?
An induced subfamily of enlargements

Let $T$ be max-mon and fix $h \in \mathcal{H}(T)$. We define

$$\tilde{T}_h : \mathbb{R}_+ \times X \Rightarrow X^*$$

as

$$\tilde{T}_h(\varepsilon, x) := \{ v \in X^* : (v, x) \in \tilde{\partial}h(2\varepsilon, x, v) \}$$

- $T_h(x) = \tilde{T}_h(0, x) = T$
- $\tilde{T}_h = L^{Ah}$, so $\tilde{T}_h \in E(T)$.

Define $E_{\mathcal{H}}(T) := \{ E \in E(T) : E = \tilde{T}_h \text{ for some } h \in \mathcal{H}(T) \}$

**Question:** $E_{\mathcal{H}}(T)$ has special properties, not shared by other elements of $E(T)$?
An induced subfamily of enlargements

Let \( T \) be max-mon and fix \( h \in \mathcal{H}(T) \). We define
\[
\tilde{\mathcal{T}}_h : \mathbb{R}_+ \times X \Rightarrow X^* \text{ as}
\]
\[
\tilde{\mathcal{T}}_h(\varepsilon, x) := \{ v \in X^* : (v, x) \in \partial\tilde{h}(2\varepsilon, x, v) \}
\]

- \( T_h(x) = \tilde{\mathcal{T}}_h(0, x) = T \)
- \( \tilde{\mathcal{T}}_h = L^{Ah} \), so \( \tilde{\mathcal{T}}_h \in \mathcal{E}(T) \).

Define \( \mathcal{E}_{\mathcal{H}}(T) := \{ E \in \mathcal{E}(T) : E = \tilde{\mathcal{T}}_h \text{ for some } h \in \mathcal{H}(T) \} \)

Question: \( \mathcal{E}_{\mathcal{H}}(T) \) has special properties, not shared by other elements of \( \mathcal{E}(T) \)?
An induced subfamily of enlargements

Let $T$ be max-mon and fix $h \in \mathcal{H}(T)$. We define
\[ \tilde{T}_h : \mathbb{R}_+ \times X \rightrightarrows X^* \]
\[ \tilde{T}_h(\varepsilon, x) := \{ v \in X^* : (v, x) \in \partial h(2\varepsilon, x, v) \} \]

- $T_h(x) = \tilde{T}_h(0, x) = T$
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Define $\mathbb{E}_{\mathcal{H}}(T) := \{ E \in \mathbb{E}(T) : E = \tilde{T}_h \text{ for some } h \in \mathcal{H}(T) \}$

Question: $\mathbb{E}_{\mathcal{H}}(T)$ has special properties, not shared by other elements of $\mathbb{E}(T)$?
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Let $T$ be max-mon and fix $h \in \mathcal{H}(T)$. We define

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- $T_h(x) = \tilde{T}_h(0, x) = T$
- $\tilde{T}_h = L^{Ah}$, so $\tilde{T}_h \in E(T)$.

Define $E_{\mathcal{H}}(T) := \{ E \in E(T) : E = \tilde{T}_h \text{ for some } h \in \mathcal{H}(T) \}$

Question: $E_{\mathcal{H}}(T)$ has special properties, not shared by other elements of $E(T)$?
Characterizing Mutual and Maximal Additivity

Let $E, E' \in \mathcal{E}(T)$, consider $h_E, h_{E'} \in \mathcal{H}(T)$ the corresponding functions (i.e., $E = L^{h_E}$ and $E' = L^{h_{E'}}$)

- $E \sim_a E'$ iff $\mathcal{J}h_E \leq h_{E'}$. Hence, $E \in \mathcal{E}_a(T)$ iff $\mathcal{J}h_E \leq h_E$.
- $h_E = \mathcal{J}h_E$ iff $E$ is max-add

- In particular, $E \sim_a L^{\mathcal{J}h_E}$
- Since $L^{\mathcal{J}h_E}$ is the largest enlargement mutually additive with $E$, it is the “additive complement” of $E$.
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Taking conjugates in $\mathcal{H}(T)$ is order reversing, and its effect in $\mathbb{E}(T)$ is to map $E$ into its additive complement.

Fixed points of $\mathcal{J}$ correspond to max-add elements!
Relation w/previous facts

Recall $\varphi^{FY}(x, v) = \varphi(x) + \varphi^*(v)$, since $J \varphi^{FY} = \varphi^{FY}$ we confirm the fact that

$\partial \varphi$ is max-add

Previous result extends the known fact (Svaiter 2000):

$T^{SE} \sim_a T^{BE}$
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Previous result extends the known fact (Svaiter 2000):

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New enlargements are additive

Let $h \in \mathcal{H}(T)$. The following hold:

1. $\tilde{T}_h \in \mathbb{E}_a(T)$

2. $\tilde{T}_h$ is max-add iff $\mathcal{J} A h = A h$

3. Hence, if $\mathcal{J} h = h$ then $\tilde{T}_h$ is max-add
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\begin{itemize}
  \item \( \tilde{T}_h \in \mathcal{E}_a(T) \)
  \item \( \tilde{T}_h \) is max-add iff \( \mathcal{J}Ah = Ah \)
  \item Hence, if \( \mathcal{J}h = h \) then \( \tilde{T}_h \) is max-add
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Fix $h \in \mathcal{H}(\partial \varphi)$ and $h \leq \varphi + \varphi^* = \varphi^{FY}$

1. $\forall \varepsilon > 0$, $x \in \text{Dom} \varphi$ we have
   $$\bar{T}_h(\varepsilon/2, x) \subseteq \bar{\partial \varphi}(\varepsilon, x)$$

2. If $h = \varphi^{FY}$ we must have $\bar{\partial \varphi} = \bar{T}_{\varphi^{FY}}$.
3. If $h = F_{\partial \varphi}$
   $$\bar{T}_{F_{\partial \varphi}}(\varepsilon/2, x) \subseteq \bar{\partial \varphi}(\varepsilon, x)$$

Hence, we can use the Fitzpatrick function $F_{\partial \varphi}$ to obtain an enlargement smaller than $\bar{\partial \varphi}$.
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- If \( h = \mathcal{F}_{\partial \varphi} \)

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Fix $h \in \mathcal{H}(\partial \varphi)$ and $h \leq \varphi + \varphi^* = \varphi^\text{FY}$

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  $$\tilde{T}_{\varepsilon} h(\varepsilon/2, x) \subseteq \tilde{\partial} \varphi(\varepsilon, x)$$

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Open problems

- Does the subfamily $E_{\mathcal{H}}(T)$ contain all additive enlargements of $T$?
- We have also seen that elements of $E_{\mathcal{H}}(T)$ are max-add when $h = \mathcal{J} h$. Are these all the max-add enlargements of $T$?
- Can we characterize $h$ such that $\mathcal{J} A h = A h$?
- In which cases has $E_{\mathcal{H}}(T)$ a single element?
- When are the max-add elements unique?
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