Lipschitz Stability in Variational Analysis

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Mathematical Reviews and the University of Michigan
1. The four theorems by Lawrence M. Graves
2. The theorems of Hildebrand-Graves and Robinson
3. Extending Robinson’s theorem
4. The theorem by A. Izmailov
5. Advertisement
6. Some open problems
The four theorems by Lawrence M. Graves

- The Hildebrand-Graves theorem (1927)
- The Lyusternik-Graves theorem (1932, 1950)
- The Bartle-Graves theorem (1952)
- The Karush-Kuhn-Tucker theorem (1939)
Hildebrand–Graves inverse function theorem (TAMS 1927)

Lipschitz modulus

$$\text{lip}(f; \bar{x}) := \limsup_{x', x \to \bar{x}, x \neq x'} \frac{\|f(x') - f(x)\|}{\|x' - x\|}.$$ 

Theorem (Hildebrandt-Graves slightly extended)

Let $X$ be a Banach space and consider a continuous function $f : X \to X$ with $f(\bar{x}) = 0$ and a linear bounded mapping $A : X \to X$ which is invertible. Suppose that

$$\text{lip}(f(\cdot) - A(\cdot - \bar{x}); \bar{x}) \cdot \|A^{-1}\| < 1.$$ 

Then the inverse $f^{-1}$ has a **single-valued localization** around 0 for $\bar{x}$ which is Lipschitz continuous near 0.
Reformulation of H-G theorem

Theorem.

Let $X$ be a Banach space and consider a continuous function $f : X \to X$ with $f(\bar{x}) = 0$ and a linear bounded mapping $A : X \to X$ such that

$$\text{lip}(f(\cdot) - A(\cdot - \bar{x}); \bar{x}) \leq \mu.$$ 

Suppose that the mapping $(A(\cdot - \bar{x}))^{-1}$ has a Lipschitz continuous single-valued localization at 0 for $\bar{x}$ with Lipschitz modulus $\leq \kappa$. Let

$$\kappa \mu < 1.$$ 

Then the inverse $f^{-1}$ has a single-valued localization around 0 for $\bar{x}$ which is Lipschitz continuous near 0.
Theorem.

Let $X$ be a Banach space and consider a continuous function $f : X \to X$ with $f(\bar{x}) = 0$ and a linear bounded mapping $A : X \to X$ such that

$$\text{lip}(f(\cdot) - A(\cdot - \bar{x}); \bar{x}) \leq \mu.$$ 

Let $G$ be any set-valued mapping from $X$ to $X$ with $0 \in G(\bar{x})$ for which the mapping $(A(\cdot - \bar{x}) + G(\cdot))^{-1}$ has a Lipschitz continuous single-valued localization around 0 for $\bar{x}$ with Lipschitz constant $\kappa$. Let

$$\kappa \mu < 1.$$ 

Then the inverse $(f + G)^{-1}$ has a single-valued localization around 0 for $\bar{x}$ which is Lipschitz continuous near 0.
The smooth Robinson’s inverse function theorem

\( F \) is **strongly metrically regular** at \( \bar{x} \) for \( \bar{y} \) if \( (\bar{x}, \bar{y}) \in \text{gph} \ F \) and \( F^{-1} \) has a **Lipschitz continuous single-valued localization** around \( \bar{y} \) for \( \bar{x} \). The Lipschitz modulus of the localization is denoted \( \text{reg}(F; \bar{x} | \bar{y}) \).

**Theorem.**

Let \( X \) be a Banach space, let \( f : X \to X \) be strictly differentiable at \( \bar{x} \) and let \( f(\bar{x}) = 0 \). Let \( F : X \rightrightarrows X \) be a set-valued mapping with \( \bar{y} \in F(\bar{x}) \). Then FAE:

1) \( Df(\bar{x})(\cdot - \bar{x}) + F \) is strongly metrically regular at \( \bar{x} \) for \( \bar{y} \);
2) \( f + F \) is strongly metrically regular at \( \bar{x} \) for \( \bar{y} \).

Moreover,

\[
\text{reg}(Df(\bar{x})(\cdot - \bar{x}) + F; \bar{x} | \bar{y}) = \text{reg}(f + F; \bar{x} | \bar{y}).
\]
A general Robinson’s theorem in metric spaces

Theorem.
Let $X$ be a complete metric space, $Y$ be a linear normed space
1) $\kappa$ and $\mu$ positive constants with $\kappa \mu < 1$.
2) $F : X \to Y$ is such that $F$ is strongly regular at $\bar{x}$ for $\bar{y}$ with $\text{reg}(F; \bar{x} | \bar{y}) \leq \kappa$.
3) $f : X \to Y$ with $f(\bar{x}) = 0$ and $\text{lip}(f; \bar{x}) \leq \mu$.
Then $f + F$ is strongly regular at $\bar{x}$ for $\bar{y}$ with

$$
\text{reg}(f + F; \bar{x} | \bar{y}) \leq (\kappa^{-1} - \mu)^{-1}.
$$
Robinson’s theorem for Newton’s Method

Newton’s method for a parameterized VI

\[ x_0 = a, \quad f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) + F(x) \ni p \]

Consider the mapping

\[ \mathbb{R}^n \times \mathbb{R}^n \ni (a, p) \mapsto \Xi(u, p) = \left\{ \{x_k\} \in l_\infty \mid x_0 = a, \right. \]

\[ f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) + F(x_{k+1}) \ni p, \quad \forall k = 1, 2, \ldots \}

**Theorem.**

The mapping \( \Xi \) has a Lipschitz continuous single-valued localization around \( (\bar{x}, 0) \) for \( \{\bar{x}\} \) if and only if \( f + F \) is strongly regular at \( \bar{x} \) for 0. Moreover, each value of the localization of \( \Xi \) is a quadratically convergent sequence.
A nonsmooth Robinson’s theorem (A. Izmailov)

**Theorem (finite dimensions).**

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be Lipschitz continuous around $\bar{x}$, let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, and let $\bar{y} \in f(\bar{x}) + F(\bar{x})$. Suppose for every $A \in \partial f(\bar{x})$ the mapping

$$G_A : x \mapsto f(\bar{x}) + A(x - \bar{x}) + F(x)$$

is strongly metrically regular at $\bar{x}$ for $\bar{y}$. Then the mapping $f + F$ is strongly metrically regular at $\bar{x}$ for $\bar{y}$.

For $F \equiv 0$ reduces to Clarke’s IFT. For $f$ smooth reduces to Robinson’s theorem.
A nonsmooth Robinson’s theorem 1 (AD and R. Cibulka)

Theorem (Banach spaces).

Let $X$ and $Y$ be Banach spaces, let $(\bar{x}, \bar{y}) \in X \times Y$, let $f : X \to Y$ and $F : X \rightrightarrows Y$ be such that $\bar{y} \in f(\bar{x}) + F(\bar{x})$. Suppose that there exist a convex subset $A$ of $\mathcal{L}(X, Y)$ and a constant $c > 0$ such that

1) there exists $r > 0$ such that for each $u$ and $v$ in $B_r(\bar{x})$ one can find $A \in A$ such that

$$\|f(v) - f(u) - A(v - u)\| \leq c\|v - u\|;$$

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2) for every $A \in \mathcal{A}$ the mapping

$$G_A : x \mapsto f(\bar{x}) + A(x - \bar{x}) + F(x)$$

is strongly metrically regular at $\bar{x}$ for $\bar{y}$; moreover, if $s_A$ is a single-valued graphical localization of $G_A^{-1}$ around $\bar{y}$ for $\bar{x}$, then

$$(c + \chi(A)) \sup_{A \in \mathcal{A}} \text{reg}(G_A; \bar{x} | \bar{y}) < 1.$$ 

Then the mapping $f + F$ is strongly metrically regular at $\bar{x}$ for $\bar{y}$.

For $F \equiv 0$ reduces to an IFT by Páles (1997).
Lemma 1. there exist constants $\alpha$, $\beta$ and $\ell$ such that for every $A \in \partial f(\bar{x})$, the mapping

$$B_{2\beta}(0) \ni y \mapsto s_A := G_A^{-1}(y) \cap B_{2\ell\beta}(\bar{x})$$

is a Lipschitz continuous function with Lipschitz constant $\ell$.

Lemma 2. For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, x' \in B_\delta(\bar{x})$ there exists $A \in \partial f(\bar{x})$ with the property

$$|f(x) - f(x') - A(x - x')| \leq \varepsilon |x - x'|.$$
Lemma 3. The function

\[ \partial f(\bar{x}) \times B_\beta(0) \ni (A, z) \mapsto \varphi(A, z) := s_A(z) \cap B_{\ell \beta}(\bar{x}) \]

has the following properties:

(a) \( \text{dom } \varphi = \partial f(\bar{x}) \times B_\beta(0) \);

(b) For each \( A \in \partial f(\bar{x}) \) the function \( \varphi(A, \cdot) = s_A \) is Lipschitz continuous on \( B_\beta(0) \) with Lipschitz constant \( \ell \);

(c) For each \( A \in \partial f(\bar{x}) \) one has \( \varphi(A, 0) = \bar{x} = s_A(0) \);

(d) \( \varphi \) is continuous in its domain.
Lemma 4. Fix \( u \in B_\delta(\bar{x}) \) and define the function

\[
\partial f(\bar{x}) \ni A \mapsto \Phi_u(A) = \varphi(A, y - f(u) + f(\bar{x}) + A(u - \bar{x})) - u.
\]

Suppose that there exist \( v \in B_\delta(\bar{x}) \setminus \{u\} \) along with \( \tilde{A} \in \partial f(\bar{x}) \) satisfying

\[
|f(v) - f(u) - \tilde{A}(v - u)| \leq \varepsilon|v - u| \quad \text{and} \quad f(v) + \tilde{A}(u - v) + F(u) \ni y.
\]

Then

\[
0 < |\Phi_u(A)| \leq \ell \varepsilon |u - v| \quad \text{whenever} \quad A \in \partial f(\bar{x}).
\]
Define the mapping

\[ K \ni h \mapsto \Psi_u(h) = \{ A \in \partial f(\bar{x}) : |f(u + h) - f(u) - Ah| \leq \varepsilon|h| \} . \]

**Lemma 5.** Given \( u \in B_\delta(\bar{x}) \), suppose that \( \Phi_u \) maps \( \partial f(\bar{x}) \) into \( K \). Then there exists a continuous selection \( \psi_u \) of the mapping \( \Psi_u \) such that the function defined as the composition \( \psi_u \circ \Phi_u \) has a fixed point in \( K \).
Lemma 6. There exist sequences \( \{x_n\} \) in \( \mathbb{R}^n \) and \( \{A_n\} \) in \( \partial f(\bar{x}) \) whose entries have the following properties for each \( n \):

(i) \( |x_n - \bar{x}| < \delta \);  
(ii) \( 0 < |x_{n+1} - x_n| \leq (l\varepsilon)^n|x_1 - x_0| \);  
(iii) \( |f(x_{n+1}) - f(x_n) - A_n(x_{n+1} - x_n)| \leq \varepsilon|x_{n+1} - x_n| \);  
(iv) \( f(x_n) + A_n(x_{n+1} - x_n) + F(x_{n+1}) \ni y \).
Lemma 7. The mapping

\[ B_{\varepsilon b}(0) \ni y \mapsto \sigma(y) := (f + F)^{-1}(y) \cap B_\delta(\bar{x}) \]

is a nonempty-valued localization of \((f + F)^{-1}\).

End of proof. The map \(\sigma\) is both single-valued and Lipschitz continuous.
Most of the results in this talk can be found in

Asen L. Dontchev · R. Tyrrell Rockafellar

Implicit Functions and Solution Mappings
A View from Variational Analysis, Second Edition

The implicit function theorem is one of the most important theorems in analysis and its many variants are basic tools in partial differential equations and numerical analysis. This second edition of Implicit Functions and Solution Mappings presents an updated and more complete picture of the field by including solutions of problems that have been solved since the first edition was published, and places old and new results in a broader perspective. The purpose of this self-contained work is to provide a reference on the topic and to provide a unified collection of a number of results which are currently scattered throughout the literature. Updates to this edition include new sections in almost all chapters, new exercises and examples, updated commentaries to chapters and an enlarged index and references section.

From reviews of the first edition:

"The book commences with a helpful context-setting preface followed by six chapters. Each chapter starts with a useful preamble and concludes with a careful and instructive commentary, while a good set of references, a notation guide, and a somewhat brief index complete this study. … I unreservedly recommended this book to all practitioners and graduate students interested in modern optimization theory or control theory or to those just engaged by beautiful analysis clearly described." (Jonathan Michael Browne, IEEE Control Systems Magazine, February, 2012)

"This book is devoted to the theory of inverse and implicit functions and some of its modifications for solution mappings in variational problems. … The book is targeted to a broad audience of researchers, teachers and graduate students. It can be used as well as a textbook as a reference book on the topic. Undoubtedly, it will be used by mathematicians dealing with functional and numerical analysis, optimization, adjacent branches and also by specialists in mechanics, physics, engineering, economics, and so on." (Peter Zabreiko, Zentralblatt MATH, Vol. 1178, 2010)

"The present monograph will be a most welcome and valuable addition. … This book will save much time and effort, both for those doing research in variational analysis and for students learning the field. This important contribution fills a gap in the existing literature." (Stephen M. Robinson, Mathematical Reviews, Issue 2010)
Some Open Problems

1. Applications of the nonsmooth Robinson’s theorem – PDE constrained optimization? (Ito and Kunish, Ulbrich)

2. Are there a nonsmooth Lyusternik-Graves theorem or a nonsmooth Bartle-Graves theorem?

3. Is there a Nash-Moser version of Robinson’s or Lyusternik-Graves theorem?

4. Find the radius of strong regularity for mappings with specific structure, e.g. the KKT system.
THANK YOU!