Semivectorial Bilevel Optimization on Riemannian Manifolds

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Outline

1. Introduction
2. Preliminary results
3. A useful equivalent form for the \((SVB_{\sigma})\)
4. Optimality Conditions
5. An existence result for the pessimistic problem
Semivectorial bilevel optimization

H. Bonnel

Introduction

Preliminary results

A useful equivalent form for the (SVB$_{\sigma}$)

Optimality Conditions

An existence result for the pessimistic problem
Semivectorial bilevel problem (SVB\(_\sigma\)): \textit{optimistic case}

**Scalar upper level (leader's objective)** \( f : \mathbb{M}_1 \times \mathbb{M}_2 \rightarrow \mathbb{R} \)

\[
\begin{align*}
\min_x \min_y f(x, y), \\
\text{subject to}
\end{align*}
\]

**Vector lower level (follower's objective)** \( F : \mathbb{M}_1 \times \mathbb{M}_2 \rightarrow \mathbb{R}' \), for each fixed \( x \in \mathbb{M}_1 \), \( y \) is a \( \sigma \)-efficient (Pareto) solution for

\[
\begin{align*}
\min_{y'} F(x, y'), \\
\text{subject to } y' \in \mathbb{M}_2.
\end{align*}
\]

Optimistic case = the followers chose a best solution for the leader among their best responses
Semivectorial bilevel problem $(SVB_\sigma)$: 
**optimistic case**

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\[ f : M_1 \times M_2 \rightarrow \mathbb{R} \]
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subject to

**Vector lower level (follower’s objective)**
\[ F : M_1 \times M_2 \rightarrow \mathbb{R}', \]
for each fixed \( x \in M_1 \), \( y \) is a \( \sigma \)-efficient (Pareto) solution for
\[ \text{MIN}_{y'} F(x, y'), \quad \text{s.t.} \quad y' \in M_2. \]

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**Optimistic case** = the followers chose a best solution for the leader among their best responses

\( M_1, M_2 \) connected Riemannian manifolds verifying Hopf-Rinow theorem; \( C \) is a closed, convex, pointed cone, \( \text{int} \ (C) \neq \emptyset \), and \( \sigma \in \{w, p\} \).
Semivectorial bilevel problem (SVB$_\sigma$): *pessimistic case*

**Pessimistic case** = *the followers may chose a worst solution for the leader among their best responses*
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Particular cases

Classical bilevel optimization (optimistic or pessimistic case)

We obtain the problem

\[
\min_x \min_y f(x, y) \quad \text{(resp. } \min_x \sup_y f(x, y)\text{)} \quad \text{subject to } \quad y \in \psi(x),
\]

where

\[
\psi(x) = \argmin_{y'} \left( F(x, y') \right) \quad \text{s.t. } \quad h(y') = 0,
\]

considering \( Z = \mathbb{R}, \ M_1 = \mathbb{R}^m, \ M_2 = \{ y \in \mathbb{R}^{n+p} \mid h(y) = 0 \} \), where \( h : \mathbb{R}^{n+p} \to \mathbb{R}^p \) is smooth and regular.
Optimization over the $\sigma$-efficient set

We obtain the problem

$$\min f(y) \quad \text{subject to}$$

$y$ is a $\sigma$-efficient solution for the vector optimization problem

$$\text{MIN}_C F(y') \quad \text{subject to} \quad y' \in S_0,$$

considering $M_1 = \{x_0\}$, $f(\cdot) = f(x_0, \cdot)$, $F(\cdot) = F(x_0, \cdot)$, $M_2 = S_0 \subset \mathbb{R}^n$. 
Some references


Some references


In the existing literature all SVB studied are considered in the Euclidean (or Hilbert, Banach) spaces.
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In my talk I consider the Riemannian setting case.
Why Riemannian?

- Constrained optimization problems can be seen as unconstrained ones from the Riemannian geometry viewpoint;
- Moreover some nonconvex optimization problems in the Euclidean setting may become convex introducing an appropriate Riemannian metric.
- In the last years researchers began the study of optimization problems in the Riemannian setting.
- Some results are new even in the Euclidean setting.
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Hopf-Rinow Theorem

Let $M$ be a connected Riemannian manifold. The following statements are equivalent:

1. $M$ is complete as a metric space.
2. $M$ is geodesically complete (i.e. all the geodesics are defined on $\mathbb{R}$).
3. Closed and bounded sets on $M$ are compact.
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Moreover, each of the statements (1-3) implies that any two points of $M$ can be joined by a minimizing geodesic.
Vector optimization on Riemannian manifolds

Let $M$ be a connected RM verifying the Hopf-Rinow theorem. Let $C \subset \mathbb{R}^r$ be a convex pointed cone, closed with $\text{int} (C) \neq \emptyset$.

For any $y, y' \in \mathbb{R}^r$ we denote

- $y \preceq y' \iff y' - y \in C$
- $y < y' \iff y' - y \in \text{int} (C)$
- $y \precsim y' \iff y' - y \in C \setminus \{0\}$.

We have

$$y < y' \implies y \precsim y' \implies y \preceq y'.$$

$\preceq$ is a partial order relation on $\mathbb{R}^r$.

$<$ and $\precsim$ are transitive relations.

\[\text{a.e. } \mathbb{R}_+C + C \subset C, \ C \cap (-C) = \{0\}\]
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\[ ^* \text{i.e. } \mathbb{R}^+ C + C \subset C, \ C \cap (-C) = \{0\} \]
Consider a vector function \( G = (G_1, \ldots, G_r) : M \rightarrow \mathbb{R}^r \), and the multiobjective optimization problem

\[
(MOP) \quad \text{MIN}_CG(x) \quad \text{s.t. } x \in M.
\]

For (MOP) the point \( a \in M \) is called:

- **Pareto solution** if there is no \( x \in M \) such that \( G(x) \npreceq G(a) \);
- **weakly Pareto solution** if there is no \( x \in M \) such that \( G(x) \prec G(a) \);
- **properly Pareto solution** if \( a \) is a Pareto solution, and there exists a pointed convex cone \( K \) such that \( C \setminus \{0\} \subset \text{int} (K) \) and \( a \) is a Pareto solution for the problem \( \text{MIN}_KG(x) \quad \text{s.t. } x \in M \), in other words \( G(M) \cap (G(a) - K) = \{G(a)\} \).
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For (MOP) the point $a \in M$ is called:

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In the particular case $C = \mathbb{R}_+^r$ (the Pareto cone), the previous definitions can be stated as follows.

- **Pareto solution** if there is no $x \in M$ such that, for all $i \in \{1, \ldots, r\}$, $G_i(x) \leq G_i(a)$, and $G(x) \neq G(a)$;
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- **properly Pareto solution** (provided that $G(M) + \mathbb{R}_+^r$ is convex) if $a$ is a Pareto solution, and there is a real number $\mu > 0$ such that for each $i \in \{1, \ldots, r\}$ and every $x \in M$ with $G_i(x) < G_i(a)$ at least one $j \in \{1, \ldots, r\}$ exists with $G_j(x) > G_j(a)$ and
  \[
  \frac{G_i(a) - G_i(x)}{G_j(x) - G_j(a)} \leq \mu.
  \]
Vector optimization on Riemannian manifolds

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\]
Vector optimization on Riemannian manifolds

In the particular case $C = \mathbb{R}^r_+$ (the Pareto cone), the previous definitions can be stated as follows.

- **Pareto solution** if there is no $x \in M$ such that, for all $i \in \{1, \ldots, r\}$, $G_i(x) \leq G_i(a)$, and $G(x) \neq G(a)$;

- **weakly Pareto solution** if there is no $x \in M$ such that, for all $i \in \{1, \ldots, r\}$, $G_i(x) < G_i(a)$;

- **properly Pareto solution** (provided that $G(M) + \mathbb{R}^r_+$ is convex) if $a$ is a Pareto solution, and there is a real number $\mu > 0$ such that for each $i \in \{1, \ldots, r\}$ and every $x \in M$ with $G_i(x) < G_i(a)$ at least one $j \in \{1, \ldots, r\}$ exists with $G_j(x) > G_j(a)$ and

\[
\frac{G_i(a) - G_i(x)}{G_j(x) - G_j(a)} \leq \mu.
\]
Vector optimization on Riemannian manifolds

Let the symbol $\sigma \in \{w, p\}$ stands for:

weak if $\sigma = w$,

or

proper if $\sigma = p$.

We denote by

$$\sigma\text{-ARGMIN}_{x \in M} C \mathcal{G}(x)$$

the set of all $\sigma$-Pareto solutions.
Vector optimization on Riemannian manifolds

Definition

A real function $h : M \rightarrow \mathbb{R}$ is called \textit{convex} if for any two distinct points $a$ and $b$ in $M$, and for any minimizing geodesic segment $\gamma : [0, 1] \rightarrow M$ with $\gamma(0) = a$, $\gamma(1) = b$, the function $h \circ \gamma$ is convex in the usual way, i.e. for all $t \in ]0, 1[$

$$h(\gamma(t)) \leq (1 - t)h(a) + th(b).$$

If the last inequality is strict, we say that $h$ is \textit{strictly convex}. 
Vector optimization on Riemannian manifolds

**Definition**

The *vector valued function* $G = (G_1, \ldots, G_r) : M \rightarrow \mathbb{R}^r$ is called *$C$-convex* (resp. *$w$-strictly $C$-convex* or *$p$-strictly $C$-convex*) if for any two distinct points $a$ and $b$ in $M$, and for any geodesic segment $\gamma : [0, 1] \rightarrow M$ with $\gamma(0) = a$, $\gamma(1) = b$, we have respectively

$\forall t \in ]0, 1[ \quad G(\gamma(t)) \lessdot (1-t)G(a)+tG(b),$

$\forall t \in ]0, 1[ \quad G(\gamma(t)) \prec (1-t)G(a)+tG(b),$

$\forall t \in ]0, 1[ \quad G(\gamma(t)) \preceq (1-t)G(a)+tG(b).$
**Remark**

In the case $C = \mathbb{R}_+^r$ we have

- $G$ is $\mathbb{R}_+^r$-convex $\iff$ $G_i$ is convex for all $i = 1, \ldots, r$;
- $G$ is w-strictly $\mathbb{R}_+^r$-convex $\iff$ $G_i$ is strictly convex for all $i = 1, \ldots, r$;
- $G$ is $\mathbb{R}_+^r$-convex and there exists $i \in \{1, \ldots, r\}$ such that $G_i$ is strictly convex $\implies$ $G$ is p-strictly $\mathbb{R}_+^r$-convex.
The **dual cone of $C$** (or positive polar cone) is the set

$$C^* := \{ \lambda \in \mathbb{R}^r | \langle \lambda, y \rangle \geq 0 \quad \forall y \in C \},$$

and its **quasi-interior** is given by

$$C^*_\# := \{ \lambda \in \mathbb{R}^r | \langle \lambda, y \rangle > 0 \quad \forall y \in C \setminus \{0\} \}.$$

Notice that

$$(\mathbb{R}_+^r)^* = \mathbb{R}_+^r, \quad (\mathbb{R}_+^r)_\# = \text{int} (\mathbb{R}_+^r) = \{ \lambda \in \mathbb{R}^r | \lambda_i > 0 \quad i = 1, \ldots, r \}.$$ 

Let us denote

$$\Lambda_\sigma = \begin{cases} 
\{ \lambda \in C^* | \| \lambda \|_1 = 1 \} & \text{if } \sigma = w \\
C^*_\# & \text{if } \sigma = p.
\end{cases}$$
Proposition

A. The dual cone $C^*$ is a closed set in $\mathbb{R}^r$.

B. The set $C^*_\#$ (the quasi-interior of $C^*$) is a nonempty open set, and it is in fact the topological interior of $C^*$.

C. The set $\Lambda_w$ is compact.

This fact it is not true in general, i.e. when $C$ is a cone in a topological vector space, but in our setting we take advantage of the finite dimension of $\mathbb{R}^r$. 
Vector optimization on Riemannian manifolds

**Theorem (Scalarization)**

For each $\sigma \in \{w, p\}$, we have

$$\bigcup_{\lambda \in \Lambda_{\sigma}} \arg\min_{x \in M} \langle \lambda, G(x) \rangle \subset \sigma\text{-}\text{ARGMIN}_C G(x).$$

Moreover, if $G$ is $C$-convex on $M$, then the previous inclusion becomes an equality, i.e.

$$\sigma\text{-}\text{ARGMIN}_C G(x) = \bigcup_{\lambda \in \Lambda_{\sigma}} \arg\min_{x \in M} \langle \lambda, G(x) \rangle.$$
Section

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Hypotheses

- Leader decision space: \((M_1, g_1)\) RM verifying HR
- Follower(s) decision space: \((M_2, g_2)\) RM verifying HR
- Leader scalar objective function: \(f : M_1 \times M_2 \to \mathbb{R}\)
- Follower(s) multiobjective function: \(F = (F_1, \ldots, F_r) : M_1 \times M_2 \to \mathbb{R}^r\)
- For each \(x \in M_1\), \(\psi(x)\) stands for the weakly or properly Pareto solution set of the follower multiobjective optimization problem, i.e.
  \[\psi(x) := \sigma\text{-ARGMIN}_{y \in M_2} C F(x, y)\]

Thus \(\psi : M_1 \Rightarrow M_2\) is a set valued function.
Hypotheses

(HC)$_\sigma$  For each $x \in M_1$,
the function $F(x, \cdot)$ is $\sigma$-strictly $C$-convex on $M_2$, $\sigma \in \{w, p\}$.

(HCC)$_\sigma$  For all $x \in M_1$ and $\lambda \in \Lambda_\sigma$, the function $y \mapsto \langle \lambda, F(x, y) \rangle$
has bounded sublevel sets, i.e., for all reals $\alpha$, the set

$$\{y \in M_2 | \langle \lambda, F(x, y) \rangle \leq \alpha\}$$

is bounded.
The problems

- The **optimistic semivectorial bilevel problem**

\[
\text{(OSB)} \quad \min_{x \in M_1} \min_{y \in \psi(x)} f(x, y).
\]

The follower cooperates with the leader, i.e., for each \(x \in M_1\), the follower chooses amongst all its \(\sigma\)-Pareto solutions (his best responses) one which is the best for the leader (assuming that such a solution exists).

- The **pessimistic semivectorial bilevel problem**

\[
\text{(PSB)} \quad \min_{x \in M_1} \sup_{y \in \psi(x)} f(x, y).
\]

There is no cooperation between the leader and the follower, and the leader expects the worst scenario, i.e., for each \(x \in M_1\), the follower may choose amongst all its \(\sigma\)-Pareto solutions (his best responses) one which is unfavorable for the leader (in this case we prefer to use “sup” instead of “max”).
The problems

- The *optimistic semivectorial bilevel problem*

\[ (OSB) \quad \min_{x \in M_1} \min_{y \in \psi(x)} f(x, y). \]

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- The *pessimistic semivectorial bilevel problem*

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An equivalent problem

**Proposition**

For any $x \in M_1$ and any $\lambda \in \Lambda_\sigma$, the real valued function

$$M_2 \ni y \mapsto \langle \lambda, F(x, y) \rangle$$

is strictly convex.

**Proposition**

For each $x \in M_1$ and $\lambda \in \Lambda_\sigma$, the minimization problem

$$\min_{y \in M_2} \langle \lambda, F(x, y) \rangle$$

admits a unique solution which will be denoted hereafter $y(x, \lambda)$. 
An equivalent problem

**Proposition**

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**Proposition**

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\]

admits a unique solution *which will be denoted hereafter* \( y(x, \lambda) \).
Corollary

For each fixed $x \in M_1$, the map $\lambda \mapsto y(x, \lambda)$ is a surjection from $\Lambda_\sigma$ to $\psi(x)$, hence

$$\psi(x) = \bigcup_{\lambda \in \Lambda_\sigma} \{y(x, \lambda)\}.$$
Theorem

Problem (OSB) is equivalent to the following problem\(^a\)

\[
\min_{x \in M_1} \min_{\lambda \in \Lambda_\sigma} f(x, y(x, \lambda))
\]

Problem (PSB) is equivalent to the following problem

\[
\min_{x \in M_1} \sup_{\lambda \in \Lambda_\sigma} f(x, y(x, \lambda))
\]

\(^a\) \(y(x, \lambda)\) is the unique solution to the problem \(\min_{y \in M_2} \langle \lambda, F(x, y) \rangle\)
1. Introduction

2. Preliminary results

3. A useful equivalent form for the (SVB_σ)

4. Optimality Conditions

5. An existence result for the pessimistic problem
From now on we suppose that $f(\cdot, \cdot)$ and $F(\cdot, \cdot)$ are smooth functions.

$\nabla_i$ stands for the gradient operator on $(M_i, g_i)$, $i = 1, 2$. $F^a$ (resp. $\lambda^a$), $a = 1, \ldots, r$, are the components functions of the map $F : M_1 \times M_2 \rightarrow \mathbb{R}^r$ (resp. the canonical coordinates of the vector $\lambda \in \mathbb{R}^r$).

**Proposition (Necessary and sufficient conditions for $y(x, \lambda)$)**

Let $\lambda \in \Lambda_\sigma$ and $x \in M_1$ be given. Let $y \in M_2$. Then

$$y = y(x, \lambda) \iff \lambda_a \nabla_2 F^a(x, y) = 0.$$
More about the map \((x, \lambda) \mapsto y(x, \lambda)\)

Consider the map \(G : \mathbb{R}^r \times M_1 \times M_2 \rightarrow TM_2\) defined by

\[
G(\lambda, x, y) = \lambda a \text{grad}_2 F^a(x, y).
\]

For each \((\lambda, x) \in \Lambda_\sigma \times M_1\), the solution \(y = y(x, \lambda)\) to the problem
\[
\min_{y \in M_2} \langle \lambda, F(x, y) \rangle
\]

satisfies the equation

\[
G(\lambda, x, y) = 0.
\]

Denote by \(\delta_2 G(\lambda, x, y) : T_y M_2 \rightarrow T_y M_2\) the partial differential of \(G\) w.r.t. to \(y\) at the point \((\lambda, x, y)\).
More about the map \((x, \lambda) \mapsto y(x, \lambda)\)

**Proposition**

Let \((\lambda_0, x_0) \in \Lambda_\sigma \times M_1\), and let \(y_0 = y(x_0, \lambda_0)\) be the unique solution of the problem \(\min_{y \in M_2} \langle \lambda_0, F(x_0, y) \rangle\).

Suppose that \(\delta_2 G(\lambda_0, x_0, y_0)\) is an isomorphism\(^a\). Then, in a neighborhood of \((\lambda_0, x_0)\) the function \(y(\cdot, \cdot)\) is smooth and

\[
\frac{\partial}{\partial \lambda} y(\lambda, x) = - (\delta_2 G(\lambda, x, y))^{-1} \circ \frac{\partial G}{\partial \lambda}(\lambda, x, y)
\]

and

\[
\delta_1 y(\lambda, x) = - (\delta_2 G(\lambda, x, y))^{-1} \circ \delta_1 G(\lambda, x, y),
\]

where \(\delta_1\) denotes the partial differential operator w.r.t. \(x \in M_1\).

\(^a\)This hypothesis holds for example if we assume that \(G\) is a conformal map, or for example if there exists a real number \(c > 0\) such that

\[
g_2(\delta_2 G(\lambda_0, x_0, y_0)(v), v) \geq cg_2(v, v), \forall v \in T_y M_2.
\]
Theorem (Necessary optimality conditions for OSB$_p$)

Suppose that \( \text{loc-arg} \min_{\lambda \in \Lambda_p} f(x, y(x, \lambda)) \neq \emptyset \) for each \( x \in M_1 \).

Let \((x^*, \lambda^*) \in M_1 \times \Lambda_p\) be a (local) solution of the problem

\[
\min_{x \in M_1} \min_{\lambda \in \Lambda_p} f(x, y(x, \lambda)).
\]

Let \( y^* = y(x^*, \lambda^*) \). Then

\[
\text{grad}_1 f(x^*, y^*) + \text{grad}_2 f(x^*, y^*) \circ \delta_1 y(x^*, \lambda^*) = 0
\]

\[
\text{grad}_2 f(x^*, y^*) \circ \frac{\partial y}{\partial \lambda}(x^*, \lambda^*) = 0.
\]

Moreover, the bilinear form \( \text{Hess}(\varphi)(x^*, \lambda^*) \) is positive semidefinite, where \((x, \lambda) \mapsto \varphi(x, \lambda) := f(x, y(x, \lambda))\).
Optimality conditions for the optimistic problem

**Theorem (Necessary optimality conditions for OSB\(_w\))**

Let \((x^*, \lambda^*) \in M_1 \times \Lambda_w\) be a (local) solution of the problem

\[
\min_{x \in M_1} \min_{\lambda \in \Lambda_w} f(x, y(x, \lambda)).
\]

Let \(y^* = y(x^*, \lambda^*)\). Then

\[
\text{grad}_1 f(x^*, y^*) + \text{grad}_2 f(x^*, y^*) \circ \delta_1 y(x^*, \lambda^*) = 0
\]

\[
\text{grad}_2 f(x^*, y^*) \circ \frac{\partial y}{\partial \lambda}(x^*, \lambda^*) + N^C_{\Lambda_w}(\lambda^*) \ni 0,
\]

where \(N^C_{\Lambda_w}(\lambda^*)\) is the Clarke normal cone to the set \(\Lambda_w\) at the point \(\lambda^*\).
Optimality conditions for the optimistic problem

For $C = \mathbb{R}^{r_+}_+$ the previous theorem becomes

**Theorem**

Let $(x^*, \lambda^*) \in M_1 \times \Lambda_w$ be a (local) solution of the problem

$$\min_{x \in M_1} \min_{\lambda \in \Lambda_w} f(x, y(x, \lambda)).$$

Let $y^* = y(x^*, \lambda^*)$. Then there exists a real $\nu$ such that

$$\nabla_1 f(x^*, y^*) + \nabla_2 f(x^*, y^*) \circ \delta_1 y(x^*, \lambda^*) = 0$$

$$\nabla_2 f(x^*, y^*) \circ \frac{\partial y}{\partial \lambda_i}(x^*, \lambda^*) = \nu \ \forall i \in I_+(\lambda^*)$$

$$\nabla_2 f(x^*, y^*) \circ \frac{\partial y}{\partial \lambda_k}(x^*, \lambda^*) \geq \nu \ \forall k \in I_0(\lambda^*),$$

where $I_+(\lambda^*) = \{i|\lambda^*_i > 0\}$ and $I_0(\lambda^*) = \{k|\lambda^*_k = 0\}$. 
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Existence result for PSB\(_w\)

For the more difficult case of the pessimistic problem we will deal only with weakly-Pareto solutions.

**Theorem**

Suppose moreover that the Riemannian manifold \((M_1, g_1)\) is compact. Then the pessimistic problem

\[
\min_{x \in M_1} \sup_{\lambda \in \Lambda_w} f(x, y(x, \lambda))
\]

has at least one global solution.
Semivectorial bilevel optimization

H. Bonnel

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THANK YOU!