Chief series of locally compact groups

Colin D. Reid (joint work with Phillip R. Wesolek)

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A **topological group** is a group that is also a topological space, such that \((x, y) \mapsto xy\) and \(x \mapsto x^{-1}\) are continuous.

**G** is **locally compact** if there is a compact neighbourhood of 1. **G** is **compactly generated** if there is a compact subset of **G** that generates **G** as a group.

Examples of compactly generated locally compact groups:

- Finitely generated groups (with the discrete topology)
- Compact groups
- Any connected locally compact group (e.g. connected subgroups of \(\text{GL}(\mathbb{R}^n)\))
- Many examples of totally disconnected locally compact groups, e.g. the automorphism group of any connected locally finite graph with finitely many orbits of vertices
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A normal factor of a topological group $G$ is a quotient $K/L$, such that $K$ and $L$ are closed normal subgroups of $G$. We say it is a chief factor if $K > L$ there does not exist $K > M > L$ such that $M$ is closed and normal in $G$.

A (finite) chief series for $G$ is a series

$$\{1\} = G_0 < G_1 < G_2 < \cdots < G_n = G$$

of closed normal subgroups of $G$, such that each $G_{i+1}/G_i$ is a chief factor.
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Every finite group has a chief series. Given any group $G$, any finite chief factor of $G$ is the product of finitely many copies of a simple group.

Connected Lie groups have something like a chief series: there is a finite series in which every factor is chief or abelian. Every non-abelian chief factor is a product of finitely many copies of a simple connected Lie group.

Compact groups have descending chief series, but these are usually infinite.

Finitely generated discrete groups can have a very complicated normal subgroup structure (e.g. a finitely generated free group), and chief series fail to capture this structure.
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Theorem 1 (Caprace–Monod 2011)
Let $G$ be a compactly generated locally compact group with no non-trivial compact or discrete normal subgroups. Then every non-trivial closed normal subgroup of $G$ contains a minimal one.

Theorem 2 (R.–Wesolek)
Let $G$ be a compactly generated locally compact group.

(i) Let $G_1 < G_2 < G_3 \ldots$ be an ascending chain of closed normal subgroups of $G$ and let $K = \bigcup_i G_i$. Then there exists $i$ such that $K/G_i$ is compact-by-discrete.

(ii) Let $G_1 > G_2 > G_3 \ldots$ be a descending chain of closed normal subgroups of $G$ and let $K = \bigcap_i G_i$. Then there exists $i$ such that $G_i/K$ is compact-by-discrete.
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Theorem 3 (R.–Wesolek)

For every compactly generated locally compact group $G$, there is an **essentially chief series**, i.e. a finite series

$$\{1\} = G_0 < G_1 < G_2 < \cdots < G_n = G$$

of closed normal subgroups of $G$, such that each $G_{i+1}/G_i$ is compact, discrete or a chief factor of $G$. 
Let $G$ be a compactly generated locally compact group. Write $G^\circ$ for the largest connected subgroup of $G$.

Fact
$G$ has an action on a graph $\Gamma$, called a Cayley-Abels graph for $G$, such that:

- $G$ acts transitively on vertices;
- The degree of $\Gamma$ (= maximum number of neighbours of a vertex) is finite;
- If $U$ is the stabiliser of a vertex, then $U$ is open in $G$ (so $G^\circ \leq U$) and $U/G^\circ$ is compact.

If $N$ is a closed normal subgroup of $G$, then $\Gamma/N$ is a Cayley-Abels graph for $G/N$. 

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Outline of proof of Theorem 2 (e.g. Theorem 2(i)):

- Fix a Cayley-Abels graph $\Gamma$ for $G$ and consider $\text{deg}(\Gamma/G_i)$. By dividing out by a large enough $G_i$, can assume $\text{deg}(\Gamma) = \text{deg}(\Gamma/K)$. Then all the vertex stabilisers in $K$ acting on $\Gamma$ are equal, so $K/N$ is a discrete group, where $N$ is the kernel of the action of $K$.

- By dividing out by a compact group, can assume $K^\circ$ is a Lie group (solution to Hilbert’s 5th problem).

- Use the structure of Lie groups to deduce that there exists $i$ such that $K^\circ/(G_i)^\circ$ is compact.

- $N$ is connected-by-compact, so $N/(G_i)^\circ$ is compact-by-compact = compact, and $K/N$ is discrete, so $K/(G_i)^\circ$ is compact-by-discrete, and hence $K/G_i$ is compact-by-discrete.
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Say the normal factors $K_1/L_1$ and $K_2/L_2$ are associated if

$$K_1L_2 = K_2L_1; \quad K_i \cap L_1L_2 = L_i \text{ for } i = 1, 2.$$ 

E.g. for any closed normal subgroups $A$ and $B$ of $G$, $A/(A \cap B)$ is associated to $AB/B$.

**Proposition (R.–Wesolek)**

For non-abelian chief factors, association is an equivalence relation. For each equivalence class, there is a canonical uppermost representative $M/C$, such that any chief factor associated to $M/C$ is of the form $A/(A \cap C)$ such that $M = AC$. In particular, there is a continuous injective homomorphism from $A/(A \cap C)$ to $M/C$ with dense image.
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Theorem 4 (R.–Wesolek)
Let $G$ be a Polish group and let

$$\{1\} = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_n = G$$

be a series of closed normal subgroups for $G$ and let $K/L$ be a non-abelian chief factor of $G$. Then there exists a unique $i$ and $G_i \leq B < A \leq G_{i+1}$ such that $A/B$ is a non-abelian chief factor associated to $K/L$. 
Say a chief factor is **non-negligible** if it is non-abelian and not associated to any compact or discrete chief factor.

**Corollary**

Given an essentially chief series

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\{1\} = G_0 < G_1 < G_2 < \cdots < G_n = G
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for the compactly generated locally compact group \( G \), then each association class of non-negligible chief factor is represented exactly once as a factor \( G_{i+1}/G_i \).

Consequently, \( G \) has only finitely many association classes of non-negligible chief factors.
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