Concentration near a hyperplane in quasi-normed spaces

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Outline

- Quasi-norms
- Small-ball estimates and structure of vectors
- Esseen inequality
- Euclidean vs non-euclidean result
- The real problem
Definition (Star-shaped domain)

A body $K \subseteq \mathbb{R}^d$ is star-shaped if $\text{conv}(\{x, 0\}) \subseteq K \ \forall x \in K$. 

Given $K \subseteq \mathbb{R}^d$ star-shaped and centrally symmetric, let $\|x\|_K = \inf\{t > 0 : x/t \in K\}$.

Definition (Quasi-norm in $\mathbb{R}^d$)

$\|\cdot\|_K$ as defined above is a quasi-norm: same as a norm but instead of the triangle inequality, $\|x + y\|_K \leq C_K(\|x\|_K + \|y\|_K)$, $C_K \geq 1$. 

Quasi-norms
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**Definition (Quasi-norm in \( \mathbb{R}^d \))**

\( \| \cdot \|_K \) as defined above is a quasi-norm, that is, same as a norm but instead of the triangle inequality,

\[
\| x + y \|_K \leq C_K (\| x \|_K + \| y \|_K), \quad C_K \geq 1.
\]

**Example: \( \ell^d_p \)**

- Take \( \mathbb{R}^d \) with \( \| x \|_p = \left( \sum_{i=1}^d |x_i|^p \right)^{1/p} \), \( p > 0 \).
- This is a quasi-norm with \( C_p = \max\{2^{1/p-1}, 1\} \) (\( \implies \) if \( p \geq 1 \) this is a norm).
- Let \( B^d_p \) be the unit ball of this (quasi-)norm.
Small-Ball Probability

- $V = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d$ a family of $n$ fixed vectors.
- $\varepsilon_1, \ldots, \varepsilon_n$ independent symmetric Bernoulli random variables.
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\varepsilon_1, \ldots, \varepsilon_n \text{ independent symmetric Bernoulli random variables.}

**Definition (Small-Ball Probability)**

Let $r > 0$, $K \subseteq \mathbb{R}^d$ symmetric star-shaped, $V$ as above. Define

$$
\rho^K_r (V) = \sup_{x \in \mathbb{R}^d} \mathbb{P}\left( \sum_{j=1}^{n} \varepsilon_j v_j \in x + rK \right).
$$
Small-Ball and structure of $V$

**Small-Ball Probability**

$$\rho^K_r(V) = \sup_{x \in \mathbb{R}^d} \mathbb{P}\left( \sum_{j=1}^n \varepsilon_j v_j \in x + rK \right).$$

$\rho^K_r(V)$ is large $\iff$ $\sum \varepsilon_j v_j$ is highly concentrated $\iff$ Much cancellation between members of $V$ $\iff$ $V$ is ‘well-structured’
Examples in $\mathbb{R}^1$

**Theorem (Erdős ’45)**

If $v_1, \ldots, v_n$ are integers, then

$$\rho_0^{B^1_2}(V) = \sup_{x \in \mathbb{R}^d} \mathbb{P}\left( \sum_{j=1}^{n} \varepsilon_j v_j = x \right) = O(n^{-1/2}).$$

**Theorem (Sárközy-Szemerédi ’65)**

If $v_1, \ldots, v_n$ are different integers, then

$$\rho_0^{B^1_2}(V) = \sup_{x \in \mathbb{R}^d} \mathbb{P}\left( \sum_{j=1}^{n} \varepsilon_j v_j = x \right) = O(n^{-3/2}).$$
Examples in $\mathbb{R}^1$

**Theorem (Erdős ’45)**

$v_1, \ldots, v_n$ integers, then

$$
\rho_{B_2^1}(V) = \sup_{x \in \mathbb{R}^d} \mathbb{P}\left( \sum_{j=1}^{n} \varepsilon_j v_j = x \right) = O(n^{-1/2}).
$$

**Theorem (Sárközy-Szemerédi ’65)**

$v_1, \ldots, v_n$ different integers, then

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$$

In general: several ways of defining ‘well-structured’.
Let $X_V$ be the random vector $\sum_{j=1}^{n} \varepsilon_j v_j$.

**Theorem (Esseen inequality '66)**

$$\rho_{B_2^d}(V) \leq \left( \frac{r}{\sqrt{d}} + \sqrt{d} \right)^d \int_{B_2^d} |\mathbb{E}(i\langle X_V, \xi \rangle)| d\xi.$$
Let $X_V$ be the random vector $\sum_{j=1}^{n} \varepsilon_j v_j$.

**Theorem (Esseen inequality ’66)**

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\rho_{r^2}^{B_2^d}(V) \leq \left( \frac{r}{\sqrt{d}} + \sqrt{d} \right)^d \int_{B_2^d} \left| \mathbb{E}(i\langle X_V, \xi \rangle) \right| d\xi.
\]

**Theorem (Esseen inequality for quasi-norms, FGG ’14)**

\[
\rho_r^K(V) \leq C_r^K d r^d \int_{\mathbb{R}^d} \left| \mathbb{E}(i\langle X_V, \xi \rangle) \right| e^{-r^2 \left\| \xi \right\|^2_2} d\xi.
\]

Using the Esseen for quasi-norms, can obtain more general versions of euclidean results.
Definition

Let $\omega_K$ be the smallest number such that $B_2^d \subseteq \omega_K K$. For example: $\omega_{B_2^d} = \omega_{B_\infty^d} = 1$, $\omega_{B_1^d} = \sqrt{d}$.

Theorem (Concentration near a hyperplane in quasi-normed space, FGG '15)

Let $\| \cdot \|_K$ be a quasi-norm on $\mathbb{R}^d$. Assume that $\ell \leq n$ is such that $\rho_r^K(V) \geq \left( \frac{C_K}{\sqrt{\ell}} \right)^d$. Then there exists a hyperplane $H$ and at least $n - \ell$ vectors from $V$ that satisfy

$$\text{dist}_K(v_j, H) = \inf_{h \in H} \| v_j - h \|_K \leq \omega_K r.$$ 

This result was proved for the euclidean norm by Tao-Vu '12.
A question from combinatorics

\[ P^K_r(d, n) = \sup_{V} \rho^K_r(V). \]

Sup over all sets of size \( n \) of vectors of length \( \geq 1 \).

Question: Estimate \( P^K_r(n, d) \).
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Theorem (Erdős ’65)

\[ \mathcal{P}_{r_1}^{B_2}(n, 1) = 2^{-n} S(n, \lfloor r \rfloor + 1). \]

\( S(n, m) \) is sum of largest \( m \) binomial coefficients.
A question from combinatorics

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Theorem (Erdős ’65)

\[ P^B^1_r(n, 1) = 2^{-n} S(n, \lfloor r \rfloor + 1). \]

\( S(n, m) \) is sum of largest \( m \) binomial coefficients.

Theorem (Frankl-Füredi ’88, Tao-Vu ’12)

\[ P^{B^d}_r(n, d) = (1 + o(1))2^{-n} S(n, \lfloor r \rfloor + 1). \]
Idea of proof (Tao-Vu)

**Theorem (Frankl-Füredi ’88, Tao-Vu ’12)**

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\mathcal{P}_{r^{B_2^d}}(d, n) = (1 + o(1))2^{-n}S(n, \lfloor r \rfloor + 1).
\]

Idea of proof (Tao-Vu)

**Lower bound:** follows from 1d result.
**Upper bound:** if the probability is too large, by the hyperplane theorem can go 1 dimension down and get a contradiction.
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Projection of euclidean ball on a hyperplane is a euclidean ball in one dimension lower. Not the case for other norms.
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**Theorem (Frankl-Füredi ’88, Tao-Vu ’12)**

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Estimating \( \mathcal{P}_r^K(d, n), K \neq B_2^d \) is still open...
The End