Moments and Densities of Short Random Walks

in all Dimensions

Jonathan M. Borwein
FRSC FAAAS FBAS FAA FAMS
Joint with Armin Straub, James Wan, (Christophe Vignat), Wadim Zudilin, ...

Director, CARMA, the University of Newcastle

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Number Theory Down Under

18-19 September, 2015
The University of Newcastle

It is our great pleasure to announce that the Centre for Computer-Assisted Research Mathematics and its Applications (CARMA) will be hosting a two-day number theory workshop **Number Theory Down Under (NTDU)** on 18th and 19th September 2015, at the University of Newcastle. The talks will be in the lecture theatre VG01. Links for the previous meetings are available here [NTDU13](#) and [NTDU14](#) and a bunch of [photographs](#).

Details about registration, confirmed participants, accommodation, travel, and anything else needed can be found below.

If you have any questions, please email [Mumtaz Hussain](mailto:Mumtaz.Hussain@newcastle.edu.au) at Mumtaz.Hussain@newcastle.edu.au.
The (complex) moment function of a 4-step walk in the plane.
• The (complex) moment function of a 4-step walk in the plane.
Outline

1. Introduction
2. Combinatorics
3. Analysis
4. Probability
5. Higher Dimensions
6. Mahler Measures
I. INTRODUCTION

• An age old question: What is a walk?
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- An age old question: What is a walk?
- Also (self-avoiding) random walks, random migrations, random flights.
Abstract

Following Pearson in 1905, we first study the expected distance and density of a two-dimensional walk in the plane with $n$ unit steps in random directions — what Pearson called a random walk. We finish by examining our prior work in higher dimensions.

- We present recent results on the densities, $p_n$, of $n$-step random uniform random walks in the plane ($d := 2\nu + 2 = 2$).
- For $n \geq 7$ asymptotic formulas first developed by Raleigh are largely sufficient to describe the density.
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- For \( 2 \leq n \leq 6 \) this is far from true, as first investigated by Pearson.
- We shall see remarkable new hypergeometric closed forms for \( p_3, p_4 \) and precise analytic information for larger \( n \).
- Heavy use is made of analytic continuation of the integral (also of modern special functions (e.g., Meijer-G) and computer algebra (CAS)).
For complex $s$

**Definition (Moment function)**

\[
W_n(s) = W_n(0; s) := \int_{[0,1]^n} \left| \sum_{k=1}^n e^{2\pi x_k i} \right|^s \, dx
\]

- $W_n$ is analytic precisely for $\Re s > -2$.
- Also, $W_n(1)$ is the *expectation*.

**Simplest** case (obvious for geometric reasons):

\[
W_1(s) = \int_0^1 \left| e^{2\pi ix} \right|^s \, dx = 1.
\]
• **Second simplest case:**

\[ W_2(1) = \int_0^1 \int_0^1 |e^{2\pi ix} + e^{2\pi iy}| \, dx \, dy = ? \]
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• There is always a 1-dimension reduction

\[
W_n(s) = \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right|^s \, d\mathbf{x}
\]

\[
= \int_{[0,1]^{n-1}} \left| 1 + \sum_{k=1}^{n-1} e^{2\pi x_k i} \right|^s \, d(x_1, \ldots, x_{n-1})
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• So \( W_2(1) = 4 \int_0^{1/2} \cos(\pi x) \, dx = \frac{4}{\pi} \).
\( n \geq 3 \) highly nontrivial and \( n \geq 5 \) still not well understood

- Similar problems often get *much* more difficult in five dimensions and above — e.g., Bessel moments, Box integrals, Ising integrals (work with Bailey, Broadhurst, Crandall, ...).

\[ W_5(1) \approx 2.0081618 \] was the best estimate we could compute directly, on 256 cores at Lawrence Berkeley Labs.

We have a general program to develop symbolic numeric techniques for multi-dimensional integrals.

Most results are published 1 (ISSAC 2011 (prize), RAMA, CMS 2012 (prize)).


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  - www.carma.newcastle.edu.au/~jb616/densities.pdf and

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Intro Comb Anal Prob Higher Dim Mahler Measures

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When the facts change, I change my mind. What do you do, sir?

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One 1500-step ramble: ...a familiar picture
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- 1D (and 3D) easy. Expectation of RMS distance is easy ($\sqrt{n}$).

...a familiar picture
One 1500-step ramble:

- 1D (and 3D) easy. Expectation of RMS distance is easy ($\sqrt{n}$).
- 1D or 2D lattice: probability one of returning to the origin.  
  
  *Drunken men get home, birds do not* (Kakutani)
1000 three-step rambles: ... a less familiar picture?
The long and the short of it
A little history — from a vast literature


R: Rayleigh gave large $n$ asymptotics:

$$p_n(x) \sim \frac{2x}{n} e^{-x^2/n} \quad (*Nature*, 1905).$$
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- **UNSW**: Donovan and Nuyens, WWII cryptography.
- Appear in quantum chemistry, in quantum physics as hexagonal and diamond lattice integers, etc …
II. COMBINATORICS

REVERSE POLISH SAUSAGE

JMB/JW

Short Random Walks
Even values are easier (combinatorial – no square roots).

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
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<tbody>
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<td>$W_2(k)$</td>
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- Can get started by *rapidly computing many values naively* as symbolic integrals.
$W_n(k)$ at even values

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\begin{array}{|c|c|c|c|c|c|c|}
\hline
k & 0 & 2 & 4 & 6 & 8 & 10 \\
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- Can get started by *rapidly* computing many values *naively* as symbolic integrals.
- Observe that $W_2(s) = \binom{s}{s/2}$ for $s > -1$.
- *MathWorld* gives $W_n(2) = n$ (trivial).
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- Entering 1, 5, 45, 545 in the *OEIS* now gives “*The function* $W_5(2n)$ (see *Borwein et al.* reference for definition).”
\( W_n(k) \) at odd integers

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*During the three years which I spent at Cambridge my time was wasted, as far as the academical studies were concerned, as completely as at Edinburgh and at school. I attempted mathematics, and even went during the summer of 1828 with a private tutor (a very dull man) to Barmouth, but I got on very slowly. The work was repugnant to me, chiefly from my not being able to see any meaning in the early steps in algebra. This impatience was very foolish, and in after years I have deeply regretted that I did not proceed far enough at least to understand something of the great leading principles of mathematics, for men thus endowed seem to have an extra sense. — Autobiography of Charles Darwin*
Resolution at even values

- Even formula counts $n$-letter abelian squares $x\pi(x)$ of length $2k$ (Shallit-Richmond (2008) give asymptotics):

\[
W_n(2k) = \sum_{a_1 + \ldots + a_n = k} \binom{k}{a_1, \ldots, a_n}^2. \quad (1)
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- **Known to satisfy convolutions**:

$$W_{n_1+n_2}(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 W_{n_1}(2j)W_{n_2}(2(k-j)), \text{ so}$$

$$W_5(2k) = \sum_j \binom{k}{j}^2 \binom{2(k-j)}{j} \sum_{\ell} \binom{j}{\ell}^2 \binom{2\ell}{\ell} = \sum_j \binom{k}{j}^2 \sum_{\ell} \binom{2(j-\ell)}{j-\ell} \binom{j}{\ell}^2 \binom{2\ell}{\ell}$$
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\]

- **and recursions** such as:

\[
(k+2)^2 W_3(2k+4) - (10k^2 + 30k + 23) W_3(2k+2) + 9(k+1)^2 W_3(2k) = 0.
\]
• \( W_{2k} \) satisfies an \( \left\lceil \frac{n+1}{2} \right\rceil \)-term recursion and \( \left\lfloor \frac{n+3}{2} \right\rfloor \) distinct iterated sums.

• Also

\[
W_3(1) = 3 \sum_{n=0}^{\infty} \binom{1/2}{n} \left( -\frac{8}{9} \right)^n \sum_{k=0}^{n} \binom{n}{k} \left( -\frac{1}{8} \right)^k \sum_{j=0}^{k} \binom{k}{j}^3
\]

\[
= 3 \sum_{n=0}^{\infty} (-1)^n \binom{1/2}{n} \sum_{k=0}^{n} \binom{n}{k} \left( -\frac{1}{9} \right)^k \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j}
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  \]
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  \]
  \item Recursion gives better approximations than many methods of numerical integration for many values of $s$.
\end{itemize}
\textbullet{} $W_n(2k)$ satisfies an $\lfloor \frac{n+1}{2} \rfloor$-term recursion and $\lfloor \frac{n+3}{2} \rfloor$ distinct iterated sums.

\textbullet{} Also

$$W_3(1) = 3 \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n \left( -\frac{8}{9} \right)^n \sum_{k=0}^{n} \binom{n}{k} \left( -\frac{1}{8} \right)^k \sum_{j=0}^{k} \binom{k}{j}^3$$

$$= 3 \sum_{n=0}^{\infty} (-1)^n \left( \frac{1}{2} \right)^n \sum_{k=0}^{n} \binom{n}{k} \left( -\frac{1}{9} \right)^k \sum_{j=0}^{k} \binom{k}{j}^2 \left( \frac{2j}{j} \right)$$

\textbullet{} \textbf{Recursion} gives better approximations than many methods of numerical integration for many values of $s$.

\textbullet{} \textbf{Tanh-sinh} (doubly-exponential) quadrature works well for $W_3$ but not so well for $W_4 \approx 1.79909248$. 
• $W_n(2k)$ satisfies an $\left\lfloor \frac{n+1}{2} \right\rfloor$-term recursion and $\left\lfloor \frac{n+3}{2} \right\rfloor$ distinct iterated sums.
• Also

$$W_3(1) = 3 \sum_{n=0}^{\infty} \binom{1/2}{n} \left(-\frac{8}{9}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left(-\frac{1}{8}\right)^k \sum_{j=0}^{k} \binom{k}{j} 3$$

$$= 3 \sum_{n=0}^{\infty} (-1)^n \binom{1/2}{n} \sum_{k=0}^{n} \binom{n}{k} \left(-\frac{1}{9}\right)^k \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j}$$

• **Recursion** gives better approximations than many methods of numerical integration for many values of $s$.
• **Tanh-sinh** (doubly-exponential) quadrature works well for $W_3$ but not so well for $W_4 \approx 1.79909248$.
• **Quasi-Monte Carlo** was *not* very accurate.
III. ANALYSIS

Visualizing $W_4$ in the complex plane
Carlson’s theorem: ...from discrete to continuous

**Theorem (Carlson (1914, PhD))**

Suppose $f(z)$ is analytic of exponential growth for $\Re(z) \geq 0$, and its growth on the imaginary axis is bounded by $e^{cy}, |c| < \pi$. If

$$0 = f(0) = f(1) = f(2) = \ldots$$

then $f(z) = 0$ identically in the region.

- **$\sin(\pi z)$** does not satisfy the conditions of the theorem, as it grows like $e^{\pi y}$ on the imaginary axis.
Carlson’s theorem: ...from discrete to continuous

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- \( \sin(\pi z) \) does not satisfy the conditions of the theorem, as it grows like \( e^{\pi y} \) on the imaginary axis.
- \( |W_n(\nu; s)| \leq n|\Re(s)| \) satisfies the conditions of the theorem (and \( W_n(0; s) \) is in fact analytic for \( \Re(s) > -2 \) when \( n > 2 \)).
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- $\sin(\pi z)$ does not satisfy the conditions of the theorem, as it grows like $e^{\pi y}$ on the imaginary axis.
- $|W_n(\nu; s)| \leq n^{\Re(s)}$ satisfies the conditions of the theorem (and $W_n(0; s)$ is in fact analytic for $\Re(s) > -2$ when $n > 2$).
- There is a lovely 1941 proof by Selberg of the bounded case.
• So integer recurrences yield complex functional equations. Viz

\[(s+4)^2W_3(s+4) - 2(5s^2 + 30s + 46)W_3(s+2) + 9(s+2)^2W_3(s) = 0.\]
Analytic continuation

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- This gives analytic continuations of the ramble integrals to the complex plane, with poles at certain negative integers (likewise for all \(n\)).

“For it is easier to supply the proof when we have previously acquired, by the method [of mechanical theorems], some knowledge of the questions than it is to find it without any previous knowledge. — Archimedes.”
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- This gives analytic continuations of the ramble integrals to the complex plane, with poles at certain negative integers (likewise for all \(n\)).

- \(W_3(s)\) has a simple pole at \(-2\) with residue \(\frac{2}{\sqrt{3\pi}}\), and other simple poles at \(-2k\) with residues a rational multiple of \(\text{Res}_{-2}\).

“For it is easier to supply the proof when we have previously acquired, by the method [of mechanical theorems], some knowledge of the questions than it is to find it without any previous knowledge. — Archimedes.”
Odd lengths look like $3$

$W_3(s)$ on $[-6, \frac{5}{2}]$

- JW proved zeroes near to but not at integers: $W_3(-2n - 1) \downarrow 0$. 
Odd lengths look like 3

\[ W_3(s) \text{ on } [-6, \frac{5}{2}] \]

- JW proved zeroes near to but not at integers: \( W_3(-2n - 1) \downarrow 0. \)
Some even lengths look more like 4

\[ L: W_4(s) \text{ on } [-6, 1/2]. \quad R: W_5 \text{ on } [-6, 2] (T), \quad W_6 \text{ on } [-6, 2] (B). \]
Some even lengths look more like $4$

- **L**: $W_4(s)$ on $[-6, 1/2]$. **R**: $W_5$ on $[-6, 2]$ (T), $W_6$ on $[-6, 2]$ (B).

- The **functional equation** (with double poles) for $n = 4$ is

$$
(s + 4)^3 W_4(s + 4) - 4(s + 3)(5s^2 + 30s + 48)W_4(s + 2) + 64(s + 2)^3 W_4(s) = 0
$$
Some even lengths look more like 4

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- **Conjecture:** multiple poles iff $4 | n$ (proven for small $n$).
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  \]

- Conjecture: multiple poles iff \( 4|n \) (proven for small \( n \)).
- Why is \( W_4 \) positive on \( \mathbb{R} \)?
A discovery demystified

In particular, we had shown that

\[
W_3(2k) = \sum_{a_1+a_2+a_3=k} \binom{k}{a_1, a_2, a_3}^2 = 3F_2 \left( \begin{array}{c}
\frac{1}{2}, -k, -k \\
1, 1
\end{array} \middle| 4 \right)
\]

where \( pF_q \) is the generalized hypergeometric function. We discovered \textit{numerically} that:

\[
V_3(1) = 1.57459 - .12602652i
\]

**Theorem (Real part (similarly in all even dimensions))**

\textit{For all integers} \( k \) \textit{we have} \( W_3(k) = \Re(V_3(k)) \).
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In particular, we had shown that

\[ W_3(2k) = \sum_{a_1+a_2+a_3=k} \binom{k}{a_1, a_2, a_3}^2 = \left( \begin{array}{c} 1/2, -k, -k \\ 1, 1 \end{array} \right)_{4} =: V_3(2k) \]

where \( pF_q \) is the generalized hypergeometric function. We discovered \textbf{numerically} that: \( V_3(1) = 1.57459 - .12602652i \)

\[ \text{Theorem (Real part (similarly in all even dimensions))} \]

\textit{For all integers } \( k \text{ we have } W_3(k) = \Re(V_3(k)). \)

\[ \text{We have a habit in writing articles published in scientific journals to make the work as finished as possible, to cover up all the tracks, to not worry about the blind alleys or describe how you had the wrong idea first. … So there isn’t any place to publish, in a dignified manner, what you actually did in order to get to do the work. -- Richard Feynman (Nobel acceptance 1966)} \]
Proof with hindsight

\( k = 1 \). From a dimension reduction, and elementary manipulations,

\[
W_3(1) = \int_0^1 \int_0^1 |1 + e^{2\pi ix} + e^{2\pi iy}| \, dx \, dy \\
= \int_0^1 \int_0^1 \sqrt{4 \sin(2\pi t) \sin(2\pi (s + t/2)) - 2 \cos(2\pi t) + 3} \, ds \, dt.
\]
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\[
W_3(1) = \int_0^1 \int_0^1 |1 + e^{2\pi ix} + e^{2\pi iy}| \, dx \, dy
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= \int_0^1 \int_0^1 \sqrt{4 \sin(2\pi t) \sin(2\pi(s + t/2)) - 2 \cos(2\pi t) + 3} \, ds \, dt.
\]

- Let $s + t/2 \to s$, and use periodicity of the integrand, to obtain

\[
W_3(1) = \int_0^1 \left\{ \int_0^1 \sqrt{4 \cos(2\pi s) \sin(\pi t) - 2 \cos(2\pi t) + 3} \, ds \right\} \, dt.
\]

The inner integral can now be computed because

\[
\int_0^\pi \sqrt{a + b \cos(s)} \, ds = 2\sqrt{a + b} \ E \left( \sqrt{\frac{2b}{a + b}} \right).
\]
Proof continued

Here $E(x)$ is the **elliptic integral** of the second kind:

$$E(x) := \int_0^{\pi/2} \sqrt{1 - x^2 \sin^2(t)} \, dx.$$ 

- After simplification,

$$W_3(1) = \frac{4}{\pi^2} \int_0^{\pi/2} (2 \sin(t) + 1) E \left( \frac{2\sqrt{2} \sin(t)}{1 + 2 \sin(t)} \right) \, dt.$$
Here $E(x)$ is the elliptic integral of the second kind:

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After simplification,

$$W_3(1) = \frac{4}{\pi^2} \int_0^{\pi/2} (2 \sin(t) + 1) E \left( \frac{2\sqrt{2\sin(t)}}{1 + 2\sin(t)} \right) dt.$$  

Now we recall Jacobi’s imaginary transform,

$$(x + 1) E \left( \frac{2\sqrt{x}}{x + 1} \right) = \Re(2E(x) - (1 - x^2)K(x))$$

and substitute. Here $K(x)$ is the elliptic integral of the first kind.
Here $E(x)$ is the **elliptic integral** of the second kind:

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and substitute. Here $K(x)$ is the **elliptic integral** of the first kind.

- This is where $\Re$ originates:
- e.g., $V_3(-1) = 0.896441 - 0.517560i$, $W_3(-1) = 0.896441$. 

Using the integral definition of $K$ and $E$, we can express $W_3$ as a double integral involving only $\sin$. Set

$$\Omega_3(a) := \frac{4}{\pi^2} \int_0^{\pi/2} \int_0^{\pi/2} \frac{1 + a^2 \sin^2(t) - 2 a^2 \sin^2(t) \sin^2(r)}{\sqrt{1 - a^2 \sin^2(t) \sin^2(r)}} \, dt \, dr,$$

so that

$$\Re(\Omega_3(2)) = W_3(1). \quad (2)$$
Proof completed

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- Expand using the binomial theorem, evaluate the integral term by term for small $a$ — where life is easier — and use analytic continuation to deduce

$$\Omega_3(2) = V_3(1). \tag{3}$$
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- As both sides satisfy the same 2-term recursion (computer provable), we are done.

QED
A pictorial ‘proof’ shows Carlson’s theorem does not apply

\[ W_3(s) - \Re V_3(s) \text{ on } [0, 12] \]
A pictorial ‘proof’ shows Carlson’s theorem does not apply

\[ W_3(s) - \Re V_3(s) \text{ on } [0, 12] \]

- This was hard to draw when discovered, as at the time we had no good closed form for \( W_3(s) \). For \( s \neq -3, -5, -7, \ldots \), we now have

\[
W_3(s) = \frac{3^{s+3/2}}{2\pi} \beta \left( s + \frac{1}{2}, s + \frac{1}{2} \right) _3F_2 \left( \frac{s+2}{2}, \frac{s+2}{2}, \frac{s+2}{2}; \frac{1}{4} \right).
\]
Closed forms

- We then **confirmed** 175 digits of

\[ W_3(1) \approx 1.57459723755189365749 \ldots \]
Closed forms

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  \[ W_3(1) \approx 1.57459723755189365749 \ldots \]

- Armed with a knowledge of elliptic integrals:

  \[ W_3(1) = \frac{16 \sqrt[3]{4 \pi^2}}{\Gamma\left(\frac{1}{3}\right)^6} + \frac{3 \Gamma\left(\frac{1}{3}\right)^6}{8 \sqrt[3]{4 \pi^4}} = W_3(-1) + \frac{6/\pi^2}{W_3(-1)}, \]

  \[ W_3(-1) = \frac{3 \Gamma\left(\frac{1}{3}\right)^6}{8 \sqrt[3]{4 \pi^4}} = \frac{2\frac{1}{3}}{4\pi^2} \beta^2 \left(\frac{1}{3}\right). \]

  Here \( \beta(s) := B(s, s) = \frac{\Gamma(s)^2}{\Gamma(2s)}. \)
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W_3(1) = \frac{16\sqrt{4\pi^2}}{\Gamma(\frac{1}{3})^6} + \frac{3\Gamma(\frac{1}{3})^6}{8\sqrt{4\pi^4}} = W_3(-1) + \frac{6/\pi^2}{W_3(-1)},
\]

\[
W_3(-1) = \frac{3\Gamma(\frac{1}{3})^6}{8\sqrt{4\pi^4}} = \frac{2\frac{1}{3}}{4\pi^2}\beta^2 \left( \frac{1}{3} \right).
\]

Here \( \beta(s) := B(s, s) = \frac{\Gamma(s)^2}{\Gamma(2s)} \).

- Obtained via singular values of the elliptic integral and Legendre’s identity.
IV. PROBABILITY

It can be readily shown that

$$P_n(r) = \int_0^\infty r J_1(ry) [J_0(y)]^n \, dy \quad (1.2)$$

where $J_k(y)$ is the Bessel function of the first kind of order $k$. Pearson tabulated $F_n(r)/2$ for $n \leq 7$, for $r$ ranging between 0 and $n$ (all that is necessary). He used a graphical procedure in getting his results, and remarked that for $n = 5$ the function appeared to be constant over the range between 0 and 1.

He states: ‘From $r = 0$ to $r = L$ (here 1) the graphical construction, however carefully reinvestigated, did not permit of our considering the curve to be anything but a straight line. . . . Even if it is not absolutely true, it exemplifies the extraordinary power of such integrals of $J$ products to give extremely close approximations to such simple forms as horizontal lines.’

Greenwood and Duncan (Reference [4]) later extended Pearson’s work for $n=6(1)24$, and more recently Scheid (Reference [5]) gave results for lower values of $n$ (2 to 6) obtained by a Monte Carlo procedure. The function $F_5(r)$ was computed for $r < 1$ on the Remington-Rand 1103 computer. The results are given in Table 1, and although the function is not constant, it differs from $1/3$ by less than .0034 in this range. This settles Pearson’s conjecture. The table given on page 51 may help investigators of Monte Carlo techniques to compare their results with the known values.

Since the function $F_5(r)$ is so nearly constant in the range between 0 and 1,
The Bessel J function

Recall, the normalized Bessel function of the first kind is

\[ j_\nu(x) = \nu! \left( \frac{2}{x} \right)^\nu J_\nu(x) = \nu! \sum_{m \geq 0} \frac{(-x^2/4)^m}{m!(m + \nu)!}. \] (4)

With this normalization, we have \( j_\nu(0) = 1 \) and

\[ j_\nu(x) \sim \frac{\nu!}{\sqrt{\pi}} \left( \frac{2}{x} \right)^{\nu+1/2} \cos \left( x - \frac{\pi}{2} \left( \nu + \frac{1}{2} \right) \right) \]

as \( x \to \infty \) on the real line.
The Bessel J function

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as \( x \to \infty \) on the real line.

- Note also that

\[ j_{1/2}(x) = \text{sinc}(x) = \frac{\sin(x)}{x} \]

- which in part explains why analysis in 3-space is so simple. More generally, all half-integer order \( j_\nu(x) \) are elementary.
Richer representations

1906. The influential Leiden mathematician J.C. Kluyver (1860-1932) — supervisor of Kloosterman — published a fundamental Bessel representation for the cumulative radial distribution function \(P_n\) and density \(p_n\):

\[
P_n(t) = t \int_0^\infty J_1(xt) J_0^n(x) \, dx
\]

\[
p_n(t) = t \int_0^\infty J_0(xt) J_0^n(x) x \, dx \quad (n \geq 4)
\]

where \(J_n(x)\) is the Bessel J function of the first kind (see Watson (1932, §49); 3-dim walks are elementary).

- From (7) below, we find

\[
p_n(1) = \text{Res}_{-2} (W_{n+1}) \quad (n = 1, 2, \ldots). \tag{6}
\]

- As \(p_2(\alpha) = \frac{2}{\pi \sqrt{4-\alpha^2}}\), we check in Maple that the following code returns \(R = 2/(\sqrt{3}\pi)\) symbolically:

\[
R := \text{identify} \left( \text{evalf}[20] \left( \int \text{BesselJ}(0, x)^3 x, x=0..\infty \right) \right)
\]
A Bessel integral for $W_n$

- Also $P_n(1) = \frac{J_0(0)^{n+1}}{n+1} = \frac{1}{n+1}$ (A question of Pearson).
A Bessel integral for $W_n$

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Integrands for $W_4(-1)$ (blue) and $W_4(1)$ (red) on $[\pi, 4\pi]$ from (8).
A Bessel integral for $W_n$

- Also $P_n(1) = \frac{J_0(0)^{n+1}}{n+1} = \frac{1}{n+1}$ (A question of Pearson).
- Broadhurst used (5) to show for $2k > s > -\frac{n}{2}$ that

$$W_n(s) = 2^{s+1-k} \frac{\Gamma(1 + \frac{s}{2})}{\Gamma(k - \frac{s}{2})} \int_0^\infty x^{2k-s-1} \left( -\frac{1}{x} \frac{d}{dx} \right)^k J_n^0(x) dx,$$

a useful oscillatory 1-dim integral (used below). Thence

$$W_n(-1) = \int_0^\infty J_n^0(x) dx, \quad W_n(1) = n \int_0^\infty J_1(x) J_0(x)^{n-1} \frac{dx}{x}. \quad (8)$$

Integrands for $W_4(-1)$ (blue) and $W_4(1)$ (red) on $[\pi, 4\pi]$ from (8).
The densities for \( n = 3, 4 \) are ‘modular’

Let \( \sigma(x) := \frac{3-x}{1+x} \). Then \( \sigma \) is an involution on \([0, 3]\) sending \([0, 1]\) to \([1, 3]\):

\[
p_3(x) = \frac{4x}{(3-x)(x+1)} p_3(\sigma(x)).
\]  

(9)

So \( \frac{3}{4} p_3'(0) = p_3(3) = \frac{\sqrt{3}}{2\pi}, p(1) = \infty \). We found:

The densities \( p_3 \) (L) and \( p_4 \) (R)
The densities for $n = 3, 4$ are ‘modular’

Let $\sigma(x) := \frac{3-x}{1+x}$. Then $\sigma$ is an involution on $[0, 3]$ sending $[0, 1]$ to $[1, 3]$:

$$p_3(x) = \frac{4x}{(3-x)(x+1)} p_3(\sigma(x)).$$

(9)

So $\frac{3}{4} p_3'(0) = p_3(3) = \frac{\sqrt{3}}{2\pi}, p(1) = \infty$. We found:

$$p_3(\alpha) = \frac{2\sqrt{3}\alpha}{\pi (3 + \alpha^2)} \binom{1}{\frac{1}{3}, \frac{2}{3}} \left( \alpha^2 \left( \frac{9 - \alpha^2}{3 + \alpha^2} \right)^2 \right) = \frac{2\sqrt{3}}{\pi} \frac{\alpha}{AG_3(3 + \alpha^2, 3 (1 - \alpha^2)^{2/3})}$$

where $AG_3$ is the cubically convergent mean iteration (1991):

$$AG_3(a, b) := \frac{a + 2b}{3} \bigotimes \left( b \cdot \frac{a^2 + ab + b^2}{3} \right)^{1/3}$$

The densities $p_3$ (L) and $p_4$ (R)
We ultimately deduce on $2 \leq \alpha \leq 4$ a hyper-closed form:

\[
p_4(\alpha) = \frac{2}{\pi^2} \frac{\sqrt{16 - \alpha^2}}{\alpha} \, {}_3F_2 \left( \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{5}{6}, \frac{7}{6} \end{array} \bigg| \frac{(16 - \alpha^2)^3}{108 \alpha^4} \right).
\]

(11)
We ultimately deduce on $2 \leq \alpha \leq 4$ a hyper-closed form:

$$p_4(\alpha) = \frac{2}{\pi^2} \frac{\sqrt{16 - \alpha^2}}{\alpha} {}_3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \mid \frac{(16 - \alpha^2)^3}{108 \alpha^4} \right).$$

(11)

$p_4$ from (11) vs 18-terms of empirical power series

Proves $p_4(2) = \frac{2^{7/3}}{3 \sqrt{3}} \Gamma \left( \frac{2}{3} \right)^{-6} = \frac{\sqrt{3}}{\pi} W_3(-1) \approx 0.494233 < \frac{1}{2}$

Empirically, quite marvelously, we found — and proved by a subtle use of distributional Mellin transforms — that on $[0, 2]$ as well:
We ultimately deduce on $2 \leq \alpha \leq 4$ a hyper-closed form:

$$p_4(\alpha) = \frac{2}{\pi^2} \frac{\sqrt{16 - \alpha^2}}{\alpha} \, _3F_2 \left( \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{5}{6}, \frac{7}{6} \end{array} \mid \frac{(16 - \alpha^2)^3}{108 \alpha^4} \right).$$ (11)

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$$p_4(\alpha) \overset{?}{=} \frac{2}{\pi^2} \frac{\sqrt{16 - \alpha^2}}{\alpha} \, _3F_2 \left( \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{5}{6}, \frac{7}{6} \end{array} \mid \frac{(16 - \alpha^2)^3}{108 \alpha^4} \right)$$ (12)

(Discovering this $\Re$ brought us full circle.)
The densities for $5 \leq n \leq 8$ (and large $n$ approximation)

- Both $p_{2n+4}, p_{2n+5}$ are $n$-times continuously differentiable for $x > 0$ ($p_n(x) \sim 2x^n e^{-x^2/n}$). So "four is small" but "eight is large."
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  \[ p_n(x) \sim \frac{2x}{n} e^{-x^2/n} \]

  So “four is small” but “eight is large.”
Indeed, PSLQ found various representations including:

\[
W_4(1) = \frac{9\pi}{4} \binom{\frac{7}{4}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1, 1, 1 \mid 1} - 2\pi \binom{\frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \mid 1}
\]

\[
= \frac{\pi}{4} \sum_{n=0}^{\infty} \frac{64(n + 1)^4 - 144(n + 1)^3 + 108(n + 1)^2 - 30(n + 1) + 3 \left(\frac{2^n}{n}\right)^6}{46^n (n + 1)^3}.
\]
An elliptic integral harvest

Indeed, PSLQ found various representations including:

\[ W_4(1) = \frac{9\pi}{4} \, {}_7F_6 \left( \begin{array}{c} 7 \ 3 \ 3 \ 3 \ 1 \ 1 \ 1 \\ 4 \ 2 \ 2 \ 2 \ 1 \ 1 \end{array} \bigg| 1 \right) - 2\pi \, {}_7F_6 \left( \begin{array}{c} 5 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \\ 4 \ 1 \ 1 \ 1 \ 1 \ 1 \end{array} \bigg| 1 \right) \]

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- Proofs rely on work by Nesterenko and by Zudilin. Inter alia:

\[ 2 \int_{0}^{1} K(k)^2 \, dk = \int_{0}^{1} K'(k)^2 \, dk = \left( \frac{\pi}{2} \right)^4 \, {}_7F_6 \left( \begin{array}{c} 5 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \\ 4 \ 1 \ 1 \ 1 \ 1 \ 1 \end{array} \bigg| 1 \right) \]
An elliptic integral harvest

Indeed, PSLQ found various representations including:

$$\begin{align*}
W_4(1) &= \frac{9\pi}{4} \ _7F_6 \left( \begin{array}{c} 7/4, 3/2, 3/2, 3/2, 1/2, 1/2, 1/2 \\ 3/4, 2, 2, 2, 1, 1 \\ \end{array} \bigg| 1 \right) - 2\pi \ _7F_6 \left( \begin{array}{c} 5/4, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2 \\ 1/4, 1, 1, 1, 1, 1 \\ \end{array} \bigg| 1 \right) \\
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\end{align*}$$
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\end{array} \right| 1 \right) - 2\pi _7F_6 \left( \begin{array}{c}
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\end{array} \right| 1 \right) 
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\frac{1}{4}, 1, 1, 1, 1, 1 \\
\end{array} \right| 1 \right). 
\]

• We also deduce that \((K', E')\) are complementary integrals

\[
W_4(-1) = \frac{8}{\pi^3} \int_0^1 K^2(k) \, dk, \quad W_4(1) = \frac{96}{\pi^3} \int_0^1 E'(k) K'(k) \, dk - 8 W_4(-1). 
\]
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Indeed, PSLQ found various representations including:

\[ W_4(1) = \frac{9\pi}{4} 7F_6 \left( \begin{array}{cccccc} \frac{7}{4}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1, 1 \end{array} | 1 \right) - 2\pi 7F_6 \left( \begin{array}{cccccc} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \end{array} | 1 \right) \]

\[ = \frac{\pi}{4} \sum_{n=0}^{\infty} \frac{64(n+1)^4 - 144(n+1)^3 + 108(n+1)^2 - 30(n+1) + 3}{(n+1)^3} \left( \frac{2n}{n} \right)^6. \]

- Proofs rely on work by Nesterenko and by Zudilin. Inter alia:

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- Much else about moments of products of elliptic integrals has been discovered (with massive 1600 relation PSLQ runs)
\( \nu := \frac{d}{2} - 1 \)

1. Even moments have a fine formula in all dimensions
   - the iterations all generalise (poles are simpler for \( d > 2 \))
V. HIGHER DIMENSIONAL WALKS

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5. Four and five step densities put up more resistance!
   - and in interesting ways
V. Radial densities for 3, 4, 5 steps in dimensions 2 to 9

For $x > 0$, $p_n(\nu; x)$ is $m$-times continuously differentiable if $m < \left(\frac{n - 1}{\nu + 1/2} - 1\right)$ (increases with $\nu$ and $n$).
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Va. Even moments

... what do they count?

Theorem (Even moments)

For all $d = 2\nu + 2$ even moments $W_n(\nu; 2k)$ are given by

$$\nu! n^{-1} (k + \nu)!(2k + \nu \cdots 2k + \nu) = k^{k+1}(k^k, \ldots, k^n)^{2k + \nu}.$$

For $n = 2$ we have $W_2(\nu; 2k) = 2^{2k + \nu} / (2k + \nu)$.

So for $\nu = 1$ and so $d = 4$, we have $W_2(1; 2k) = C_{k+1}$, the Catalan number of order $k + 1$.

More generally $W_n(\nu, 2k)$ is only fully integral for $\nu = 0, 1$. Indeed ...
Theorem (Even moments)

For all $d = 2\nu + 2$ even moments $W_n(\nu; 2k)$ are given by

$$
\nu!n^{-1} \frac{(k + \nu)!}{(k + n\nu)!} \sum_{k_1 + \ldots + k_n = k} \binom{k}{k_1, \ldots, k_n} \binom{k + n\nu}{k_1 + \nu, \ldots, k_n + \nu}.
$$
Theorem (Even moments)

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\nu!^{n-1} \frac{(k + \nu)!}{(k + n\nu)!} \sum_{k_1 + \ldots + k_n = k} \binom{k}{k_1, \ldots, k_n} \binom{k + n\nu}{k_1 + \nu, \ldots, k_n + \nu}.
\]

For \( n = 2 \) we have

\[
W_2(\nu; 2k) = \frac{2(k+\nu)}{\binom{k+\nu}{\nu}}.
\]

So for \( \nu = 1 \) and so \( d = 4 \), we have

\[
W_2(1; 2k) = C_{k+1},
\]

the **Catalan number** of order \( k + 1 \). More generally \( W_n(\nu, 2k) \) is only fully integral for \( \nu = 0, 1 \). Indeed ...
Theorem (BSV, 2015)

For given integer $\nu \geq 0$, let $A(\nu)$ be the infinite lower triangular matrix with entries

$$A_{k,j}(\nu) := \binom{k}{j} \frac{(k + \nu)!\nu!}{(k - j + \nu)!(j + \nu)!}$$

for row indices $k = 0, 1, 2, \ldots$ and columns $j = 0, 1, 2, \ldots$. Then the moments $W_{n+1}(\nu; 2k)$ are given by the row sums of $A(\nu)^n$. 
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- $A(1)$ is the integral Narayana triangle [A001263].
Va. Narayana Triangle

\[ A(1) := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
1 & 6 & 6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
1 & 10 & 20 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & \\
1 & 15 & 50 & 50 & 15 & 1 & 0 & 0 & 0 & 0 & \\
1 & 21 & 105 & 175 & 105 & 21 & 1 & 0 & 0 & 0 & \\
1 & 28 & 196 & 490 & 490 & 196 & 28 & 1 & 0 & 0 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix} \]
For integer $\nu \geq 0$, H&M (2015) define

$$r_{\nu} := \min \left\{ r > 0 : A_{k,j}(\nu) \in \frac{1}{r} \mathbb{Z}, j, k \geq 0 \right\}.$$ 

so that $r_0 = r_1 = 1$ and $r_2 = 3$.

**Theorem**

For $\nu \geq 1$ we have $r_{\nu} \mid \frac{(2\nu-1)!}{\nu!}$. 
For integer \( \nu \geq 0 \), H&M (2015) define

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**Theorem**

For $\nu \geq 1$ we have $\binom{2\nu-1}{\nu} \mid r_{\nu}$.

**Conjecture (Proven for $\nu = 0, 1, 2, 3, 4$)**

For $\nu \geq 1$ we have $r_{\nu} = \binom{2\nu-1}{\nu}$. 

... also congruences
Meijer-G functions (1936– )

Definition (Meijer-G)

\[ G_{p,q}^{m,n} \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \Big| x \right) := \frac{1}{2\pi i} \times \]

\[ \int_L \prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s) \frac{1}{\prod_{j=n+1}^{p} \Gamma(a_j - s) \prod_{j=m+1}^{q} \Gamma(1 - b_j + s)} x^s ds. \]
Definition (Meijer-G)

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G_{m,n}^{p,q} \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \bigg| x \right) := \frac{1}{2\pi i} \times \int_L \prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s) \prod_{j=m+1}^{q} \Gamma(1 - b_j + s) x^s ds.
\]

- **Contour** \( L \) chosen so it lies between poles of \( \Gamma(1 - a_i - s) \) and of \( \Gamma(b_i + s) \).
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- A broad generalization of hypergeometric functions — capturing Bessel \( Y, K \) and much more.
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Definition (Meijer-G)

\[ G_{p,q}^{m,n} \left( a_1, \ldots, a_p \mid b_1, \ldots, b_q \mid x \right) := \frac{1}{2\pi i} \times \]
\[ \int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=n+1}^p \Gamma(a_j - s) \prod_{j=m+1}^q \Gamma(1 - b_j + s)} x^s \, ds. \]

- **Contour** \( L \) chosen so it lies between poles of \( \Gamma(1 - a_i - s) \) and of \( \Gamma(b_i + s) \).
- A broad generalization of hypergeometric functions — capturing Bessel \( Y, K \) and much more.
- Important in CAS, they often lead to superpositions of hypergeometric terms.
Theorem (Meijer forms)

For all complex $s$, and $\nu = 0, 1/2, 1, \ldots$, with some restriction on $s$, we have

$$W_3(\nu; s) = 2^{2\nu} \nu!^2 \frac{\Gamma \left( \frac{s}{2} + \nu + 1 \right)}{\Gamma \left( \frac{1}{2} \right) \Gamma \left( -\frac{s}{2} \right)} G_{3,3}^{2,1} \left( \begin{array}{c} 1, 1 + \nu, 1 + 2\nu \\ \frac{1}{2} + \nu, -\frac{s}{2}, -\frac{s}{2} - \nu \end{array} ; \frac{1}{4} \right).$$
Theorem (Meijer forms)

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1, 1 + \nu, 1 + 2\nu \\
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\end{array} ; \frac{1}{4} \right),
\]

\[
W_4(\nu; s) = 2^{s+4\nu} \nu!^3 \frac{\Gamma \left( \frac{s}{2} + \nu + 1 \right)}{\Gamma \left( \frac{1}{2} \right)^2 \Gamma \left( -\frac{s}{2} \right)} G_{4,4}^{2,2} \left( \begin{array}{c}
1, \frac{1-s}{2} - \nu, 1 + \nu, 1 + 2\nu \\
\frac{1}{2} + \nu, -\frac{s}{2}, -\frac{s}{2} - \nu, -\frac{s}{2} - 2\nu
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**Theorem (Meijer forms)**

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\]

\[
W_4(\nu; s) = 2^{s+4\nu} \nu!^3 \frac{\Gamma \left( \frac{s}{2} + \nu + 1 \right)}{\Gamma \left( \frac{1}{2} \right)^2 \Gamma \left( -\frac{s}{2} \right)} \times G^{2,2}_{4,4} \left( \frac{1, 1-s}{\frac{1}{2} + \nu, -\frac{s}{2}, -\frac{s}{2} - \nu, -\frac{s}{2} - 2\nu}; 1 \right).
\]

- These can be written in terms of hypergeometric functions; in the limit for odd integers.
Vc. Density in odd dimensions

Theorem (Convolution for density in odd dim., García-Pelayo 2012)

Assume the dimension \( d = 2m + 1 \) is odd. Then, for all real \( x \),

\[
p_n(m - 1/2; x) = \frac{(2x)^{2m} \Gamma(m)}{\Gamma(2m)} \left( -\frac{1}{2x} \frac{d}{dx} \right)^m P_{m,n}(x)
\]

where \( P_{m,n} \) is the piecewise polynomial obtained from convolving

\[
f_m(x) := \frac{\Gamma(m + 1/2)}{\Gamma(1/2)\Gamma(m)} \begin{cases} (1 - x^2)^{m-1} & \text{if } x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}
\]

\( n - 1 \) times with itself.
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\]

\( n - 1 \) times with itself.

- The expression above is both elegant and compact. It shows that in odd dimensions the density is piecewise polynomial, but is difficult to manipulate or compute with or without a CAS. It leads to ...

... and hence also moments
Vc. Density in odd dimensions

Theorem (Densities in odd dimensions, B–Sinnamon 2015)

Let \( n \geq 2 \) and \( m \geq 1 \). Then

\[
p_n(m - 1/2; x) = \left( \frac{\Gamma(2m)}{2^m \Gamma(m)} \right)^n \sum_{r=0}^{n} \binom{n}{r} (-1)^{mr} H(n - 2r + x) \]

\[
\times \sum_{k=1}^{m} (-2)^k \binom{m - 1}{k - 1} \frac{(2m - 1 - k)!}{(2m - 1)!} x^k
\]

\[
\times \sum_{j=0}^{(m-1)n} \frac{(n - 2r + x)^{mn-1+j-k}}{(mn - 1 + j - k)!} [x^j] C_m(x)^r C_m(-x)^{n-r} \quad (14)
\]

where \( H(x) \) is the Heaviside function and

\[
C_m(x) := \sum_{k=0}^{m-1} \frac{(m - 1 + k)!}{2^k k!(m - 1 - k)!} x^k = \, _2F_0 \left( m, 1 - m; \; ; -\frac{x}{2} \right).
\]
Vd. Moments of a three step walk in even dimensions

**Theorem (Three step moments)**

For all integers $\nu$ and $n$ we have

$$W_3(\nu, n) = \text{Re} \, _3F_2 \left( \begin{array}{c} \nu + \frac{1}{2}, -n, -n - \nu \\ \nu + 1, 2\nu + 1 \end{array} \left| 4 \right. \right),$$

and, all these lie in the vector space over $\mathbb{Q}$ generated by

$$A := \frac{3}{16} \frac{2^{1/3}}{\pi^4} \Gamma^6 \left( \frac{1}{3} \right) \quad \text{and} \quad \frac{1}{\pi^2 A}.$$
Vd. Moments of a three step walk in even dimensions

Theorem (Three step moments)

For all integers $\nu$ and $n$ we have

$$W_3(\nu, n) = \operatorname{Re}_3 F_2 \left( \begin{array}{c} \nu + \frac{1}{2}, -n, -n - \nu \\ \nu + 1, 2\nu + 1 \end{array} \bigg| 4 \right),$$

and, all these lie in the vector space over $\mathbb{Q}$ generated by

$$A := \frac{3}{16} \cdot \frac{2^{1/3}}{\pi^4} \Gamma^6 \left( \frac{1}{3} \right) \quad \text{and} \quad \frac{1}{\pi^2 A}.$$

This relies on discovery that

$$W_3(\nu; 2n - 1) = 2\nu^2 \frac{W_3(\nu - 1; 2n + 3) - 3 W_3(\nu - 1; 2n + 1)}{(2n + 6\nu - 1)(2n + 1)} \quad \text{(15)}$$

• Theorem fails in odd dim but (15) has a partner for $n = 4$ yielding all odd moments of 4-step walks in even dimensions.
Vd. Moments of a three step walk in even dimensions

Theorem (OGF for even moments with 3 steps)

For integers $\nu \geq 0$ we have

$$\sum_{k=0}^{\infty} W_3(\nu; 2k)x^k = \left(\frac{(-1)^\nu}{\binom{2\nu}{\nu}}\right) \frac{(1 - 1/x)^{2\nu}}{1 + 3x} \frac{27x(1 - x)^2}{(1 + 3x)^3} _2F_1 \left(\frac{1}{3}, \frac{2}{3}; 1 + \nu \left| \frac{27x(1 - x)^2}{(1 + 3x)^3} \right. \right)$$

$$-q_\nu \left(\frac{1}{x}\right),$$

(16)

for $|x| < 1/9$, where $q_\nu(x)$ is a polynomial (that is, $q_\nu(1/x)$ is the principal part of the hypergeometric term on the right-hand side).
Vd. Moments of a three step walk in even dimensions

Theorem (OGF for even moments with 3 steps)

For integers \( \nu \geq 0 \) we have

\[
\sum_{k=0}^{\infty} W_3(\nu; 2k)x^k = \frac{(-1)^\nu}{\binom{2\nu}{\nu}} \frac{(1 - 1/x)^{2\nu}}{1 + 3x} \left( \frac{1}{3}, \frac{2}{3} \right) _2 F_1 \left( \frac{1}{3}, \frac{2}{3} \left| \frac{27x(1 - x)^2}{(1 + 3x)^3} \right. \right) \\
- q_\nu \left( \frac{1}{x} \right),
\]

for \( |x| < 1/9 \), where \( q_\nu(x) \) is a polynomial (that is, \( q_\nu(1/x) \) is the principal part of the hypergeometric term on the right-hand side).

- \( q_0(x) = 0 \) and \( q_1(x) = \frac{1}{2x^2} - \frac{1}{x} \), etc.
Vd. Density of a three step walk in all dimensions

**Theorem (Three step density)**

For any half-integer $\nu$ and $x \in (0, 3)$, we have

$$
\frac{p_3(\nu; x)}{x} = \frac{2\sqrt{3}}{\pi} \frac{3^{-3\nu}}{(2\nu)} \frac{x^{2\nu}(9 - x^2)^{2\nu}}{3 + x^2} \; _2F_1 \left( \frac{1}{3}, \frac{2}{3} ; \frac{x^2(9 - x^2)^2}{(3 + x^2)^3} \right). \tag{17}
$$

In addition, $p_3(\nu; x)/x$ satisfies the functional equation

$$
F(x) = \left( \frac{1 + x}{2} \right)^{6\nu + 2} F \left( \frac{3 - x}{1 + x} \right).
$$

found symbolically in odd dimensions.
Vd. Density of a three step walk in all dimensions

Theorem (Three step density)

For any half-integer $\nu$ and $x \in (0, 3)$, we have

$$p_3(\nu; x) = \frac{2\sqrt{3}}{\pi} \frac{3^{-3\nu}}{\binom{2\nu}{\nu}} \frac{x^{2\nu}(9 - x^2)^{2\nu}}{3 + x^2} 2F_1 \left( \frac{1}{3}, \frac{2}{3}; 1 + \nu; \frac{x^2(9 - x^2)^2}{(3 + x^2)^3} \right).$$

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In addition, $p_3(\nu; x)/x$ satisfies the functional equation

$$F(x) = \left( \frac{1 + x}{2} \right)^{6\nu+2} F \left( \frac{3 - x}{1 + x} \right).$$

found symbolically in odd dimensions.

General results for all $n = 3, 4, 5 \ldots$ and $\nu > 0$ include:

$$p_{n+1}^{(d-1)}(\nu; 0) = (d - 1)! p_n(\nu; 1),$$

$$p'_n(\nu; 1) = \frac{(2n)^\nu + n - 1}{n + 1} p_n(\nu; 1).$$
Ve. Generalised Domb numbers

We can prove $W_4(\nu; 2k) =$

$$2^{2(\nu+k)} \frac{\Gamma \left( k + \nu + \frac{1}{2} \right) \Gamma \left( 1 + \nu \right)}{\sqrt{\pi} \Gamma \left( 1 + k + 2\nu \right)} 3F_2 \left( \begin{array}{c} -k, -k - \nu, -k - 2\nu, \frac{1}{2} + \nu \\ 1 + \nu, 1 + 2\nu, \frac{1}{2} - k - \nu \end{array} \right| 1 \right).$$

(18)

...and an OGF for (19)
We can prove \( W_4(\nu; 2k) = \)

\[
2^{2(\nu+k)} \frac{\Gamma \left( k + \nu + \frac{1}{2} \right) \Gamma (1 + \nu)}{\sqrt{\pi} \Gamma (1 + k + 2 \nu)} 3F2 \left( \begin{array}{c} -k, -k - \nu, -k - 2\nu, \frac{1}{2} + \nu \\ 1 + \nu, 1 + 2\nu, \frac{1}{2} - k - \nu \end{array} \bigg| 1 \right).
\]

(18)

The Domb or diamond lattice numbers start: 1, 4, 28, 256, 2716, 31504, 387136, 4951552.... They are A002895 in the OEIS with ogf

\[
1 + 4 x^2 + 28 x^4 + \ldots = \frac{1}{1 - 4 x} \left( \frac{1}{6}, \frac{1}{3} \right| \frac{108 x^2}{(1 - 4 x)^3} \right)^2.
\]
Ve. Generalised Domb numbers

We can prove $W_4(\nu; 2k) =$

$$2^{2(\nu+k)} \frac{\Gamma(k + \nu + \frac{1}{2}) \Gamma(1 + \nu)}{\sqrt{\pi} \Gamma(1 + k + 2\nu)} \cdot 3F_2 \left( \begin{array}{c} -k, -k - \nu, -k - 2\nu, \frac{1}{2} + \nu \\ 1 + \nu, 1 + 2\nu, \frac{1}{2} - k - \nu & 1 \end{array} \right).$$

(18)

The Domb or *diamond lattice numbers* start: 1, 4, 28, 256, 2716, 31504, 387136, 4951552.... They are A002895 in the OEIS with ogf

$$1 + 4x^2 + 28x^4 + \ldots = \frac{1}{1 - 4x} 2F_1 \left( \begin{array}{c} \frac{1}{6}, \frac{1}{3} \\ 1 \end{array} \left| \frac{108x^2}{(1 - 4x)^3} \right. \right)^2.$$

- For 4-steps in $d = 4, 6$ dim. (18) gives [A253095, 14-06-15]

  $$1, 4, 22, 148, 1144, 9784, 90346, 885868, 9115276, \ldots \quad (19)$$

  $$1, 4, 20, \frac{352}{3}, \frac{2330}{3}, \frac{16952}{3}, \frac{133084}{3}, 370752, 3265208, \ldots \quad (20)$$

  which is what the Narayana analysis showed.
It was known that

$$\sum_{k=0}^{\infty} W_4(0; 2k)x^k = \frac{1}{1 - 16x} 2F_1 \left( \begin{array}{c} \frac{1}{6}, \frac{1}{3} \\ 1 \end{array} \left| \frac{108x}{(16x - 1)^3} \right. \right)^2. \quad (21)$$
ve. Generalised Domb numbers

...and an ogf for (19)

It was known that

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We derived, as in (16), that

\[ \frac{1}{2x^2} + \frac{1}{x} + \sum_{n=0}^{\infty} W_4(1; 2k)x^k \]

\[ = (32x - 7)F_0^2 - (4x - 1) \left[ (32x + 3)F_0F_1 - \left( 16x^2 + 10x + \frac{1}{4} \right) F_1^2 \right]. \] (22)

Here, we employ hypergeometrics:

\[ F_\lambda := \frac{1}{2 \cdot 3^\lambda x(16x - 1)^{1-\lambda}} \frac{d^\lambda}{dx^\lambda} 2F_1 \left( \begin{array}{c} \frac{1}{6}, \frac{1}{3} \\ 1 \end{array} \left| \frac{108x}{(16x - 1)^3} \right. \right). \]
• The functional equation for $W_5 = W_5(0; \cdot)$ is:

$$225(s + 4)^2(s + 2)^2W_5(s) = -(35(s + 5)^4 + 42(s + 5)^2 + 3)W_5(s + 4)$$
$$+ (s + 6)^4W_5(s + 6) + (s + 4)^2(259(s + 4)^2 + 104)W_5(s + 2).$$  \quad (23)$$

Q. Is there a hyper-closed form for $W_5(\pm 1)$?
Vf. Five step walks ... now extended to all dimensions

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- We deduce the first two poles — and so all — are simple since

$$\lim_{s \to -2} (s + 2)^2W_5(s) = \frac{4}{225} (285W_5(0) - 201W_5(2) + 16W_5(4)) = 0$$
$$\lim_{s \to -4} (s + 4)^2W_5(s) = -\frac{4}{225} (5W_5(0) - W_5(2)) = 0.$$

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- We stumbled upon a proof, via Chowla-Selberg, that

  $r_{5,0} = p_4(1) = \text{Res}_{-2}(W_5) = \frac{\sqrt{15}}{3\pi} \binom{3}{\frac{1}{3}, \frac{2}{3}, \frac{1}{2}}_{-4}$.
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\]

\[
= \frac{\sqrt{15}}{3\pi} 3F_2 \left( \frac{1}{3}, \frac{2}{3}, \frac{1}{2} \mid -4 \right).
\]

- And, originally numerically, but now proven using $\nu = 1$, that

\[
r_{5,1} := \text{Res}_{-4}(W_5) = \frac{13}{225} r_5(1) - \frac{2}{5\pi^4} \frac{1}{r_5(1)}.
\]

Q. Is there a hyper-closed form for $W_5(\mp 1)$?
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$$

Q. Is there a hyper-closed form for $W_5(\mp 1)$?
We obtain three differential relations for $p_5$. Assisted by Koutschan’s HolonomicFunctions package, we computed a Gröbner basis for the ideal that they generate. From that, we find there exists, in analogy with four steps, a relation

$$x^2 p_5(\nu + 1; x) = A p_5(\nu; x) + B p_5'(\nu; x) + C p_5''(\nu; x) + D p_5'''(\nu; x),$$

with $A, B, C, D$ polynomials of degrees 12, 13, 14, 15 in $x$ (with coefficients that are rational functions in $\nu$).
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- We conclude inductively that, for integers $\nu$, the density $p_5(\nu; x)$ has a Taylor expansion at $x = 0$ whose Taylor coefficients are recursively computable and lie in the $\mathbb{Q}$-span of

$$r_{5,0} = \frac{\sqrt{5}}{40\pi^4} \Gamma \left( \frac{1}{15} \right) \Gamma \left( \frac{2}{15} \right) \Gamma \left( \frac{4}{15} \right) \Gamma \left( \frac{8}{15} \right)$$

and $1/\left(\pi^4 r_{5,0}\right)$.
We obtain three differential relations for \( p_5 \). Assisted by Koutschan’s HolonomicFunctions package, we computed a Gröbner basis for the ideal that they generate. From that, we find there exists, in analogy with four steps, a relation

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x^2 p_5(\nu + 1; x) = A p_5(\nu; x) + B p'_5(\nu; x) + C p''_5(\nu; x) + D p'''_5(\nu; x),
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with \( A, B, C, D \) polynomials of degrees 12, 13, 14, 15 in \( x \) (with coefficients that are rational functions in \( \nu \)).

- We conclude inductively that, for integers \( \nu \), the density \( p_5(\nu; x) \) has a Taylor expansion at \( x = 0 \) whose Taylor coefficients are recursively computable and lie in the \( \mathbb{Q} \)-span of

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\]

and \( 1/(\pi^4 r_{5,0}) \).

- It remains an open challenge, including in the planar case, to obtain a more explicit description of \( p_5(\nu; x) \).
The poles of $W_5$ are simple, so no logarithmic terms are involved in $p_5(\nu, x)$. Computing a few more residues from the recursion (23), near 0 we have

$$p_5(0; x) = 0.329934 x + 0.006616 x^3 + 0.00026 x^5 + 0.000014 x^7 + O(x^9)$$

(with each coefficient given to six digits of precision only), explaining the strikingly straight shape of $p_5(0; x)$ on $[0, 1]$. 

Figure: The series (dotted) and $p_5(0; x)$. 

... Pearson explained
Tantalizing parallels link the ODE methods we used for $p_4$ to those for the logarithmic Mahler measure of a polynomial $P$ in $n$-space:

$$
\mu(P) := \int_0^1 \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n})| \, d\theta_1 \cdots d\theta_n.
$$

Indeed

$$
\mu(1 + n - 1 \sum_{k=1}^n x_k) = W_n'(0).
$$

(24)

which we have evaluated in for $n = 3$ and $n = 4$ respectively in terms of log-sine integrals.

• $\mu(P)$ turns out to be an example of a period. When $n = 1$ and $P$ has integer coefficients $\exp(\mu(P))$ is an algebraic integer. In several dimensions life is harder.

• There are remarkable recent results — many more discovered than proven — expressing $\mu(P)$ arithmetically.
Tantalizing parallels link the ODE methods we used for $p_4$ to those for the logarithmic Mahler measure of a polynomial $P$ in $n$-space:

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Indeed

$$\mu \left( 1 + \sum_{k=1}^{n-1} x_k \right) = W'_n(0).$$ (24)

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VI. OPEN PROBLEMS (Mahler measures, I)

Tantalizing parallels link the ODE methods we used for $p_4$ to those for the logarithmic Mahler measure of a polynomial $P$ in $n$-space:

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- $\mu(P)$ turns out to be an example of a period. When $n = 1$ and $P$ has integer coefficients $\exp(\mu(P))$ is an algebraic integer. In several dimensions life is harder.
- There are remarkable recent results — many more discovered than proven — expressing $\mu(P)$ arithmetically.
Open problems (Mahler measures, II)

- \( \mu(1 + x + y) = L_3'(-1) = \frac{1}{\pi} \text{Cl} \left( \frac{\pi}{3} \right) \) (Smyth).
- \( \mu(1 + x + y + z) = 14 \zeta'(-2) = \frac{7}{2} \frac{\zeta(3)}{\pi^2} \) (Smyth).
Open problems (Mahler measures, II)

- $\mu(1 + x + y) = L'_3(-1) = \frac{1}{\pi} \text{Cl} \left( \frac{\pi}{3} \right)$ (Smyth).
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- (24) recaptures both Smyth’s results.
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- (24) recaptures both Smyth’s results.
- Deninger’s 1997 conjecture, proven by Rogers-Zudilin, is

\[
\mu(1 + x + y + 1/x + 1/y) \overset{?}{=} \frac{15}{4\pi^2} L_E(2)
\]

— an L-series value for an elliptic curve \( E \) with conductor 15.
Open problems (Mahler measures, II)

- $\mu(1 + x + y) = L_3'(-1) = \frac{1}{\pi} \Cl \left( \frac{\pi}{3} \right)$ (Smyth).
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- Deninger’s 1997 conjecture, proven by Rogers-Zudilin, is

$$\mu(1 + x + y + 1/x + 1/y) = \frac{15}{4 \pi^2} L_E(2)$$

— an L-series value for an elliptic curve $E$ with conductor 15.
- Similarly for (24) ($n = 5, 6$) conjectures of Villegas become:

$$W_5'(0) = \left( \frac{15}{4 \pi^2} \right)^{5/2} \int_0^\infty \left\{ \eta^3(e^{-3t})\eta^3(e^{-5t}) + \eta^3(e^{-t})\eta^3(e^{-15t}) \right\} t^3 \, dt$$

$$W_6'(0) = \left( \frac{3}{\pi^2} \right)^{3} \int_0^\infty \eta^2(e^{-t})\eta^2(e^{-2t})\eta^2(e^{-3t})\eta^2(e^{-6t}) t^4 \, dt$$

using Dedekind’s $\eta$: $\eta(q) := q^{1/24} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}$. 

JMB/JW Short Random Walks
Thank you ...

My younger collaborators (2010)
Conclusion. We continue to be fascinated by this blend of combinatorics, number theory, analysis, probability, and differential equations, all tied together with experimental mathematics.

My younger collaborators (2010)