A CLOSED FORM FOR THE DENSITY FUNCTIONS OF RANDOM WALKS IN ODD DIMENSIONS

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Abstract

We derive an explicit piecewise-polynomial closed form for the probability density function of the distance traveled by a uniform random walk in an odd-dimensional space, based on recent work of Borwein, Straub, and Vignat [1] and by R. García-Pelayo [3].

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1. Preliminaries

In [1], the authors explore the distance traveled by a uniform \( n \)-step random walk in \( \mathbb{R}^d \) with unit step length. Following their lead, we denote the probability density function of this distance by \( p_n(m - 1/2; x) \), where \( m = \frac{d-1}{2} \).

We recall that the density can be expressed in terms of an integral engaging the normalized Bessel function of the first kind of order \( \nu \), defined by

\[
 j_\nu(x) = \nu! \left( \frac{2}{x} \right)^\nu J_\nu(x) = \nu! \sum_{k \geq 0} \frac{(-x^2/4)^k}{k!(k + \nu)!},
\]

(1.1)

With this normalization, we have \( j_\nu(0) = 1 \) and obtain:

**Theorem 1 (Bessel representation [1, 4]).** The probability density function of the distance to the origin in \( d \geq 2 \) dimensions after \( n \geq 2 \) steps is, for \( x > 0 \),

\[
 p_n(m - 1/2; x) = \frac{2^{-m+1/2}}{\Gamma(m + 1/2)} \int_0^\infty (tx)^{m+1/2} J_{m-1/2}(tx) j_\nu^m(t) \, dt,
\]

(1.2)

wherein \( m = \frac{d-1}{2} \).

The study of the density \( p_n(\nu; x) \) is quite classical, originating in the early 20th century [2, 4–7]. The most fundamental cases are that of two dimensions [2] and three dimensions [7]. The Bessel representation of the density is valuable for its generality and its analytically-pleasing structure, which form the basis for many related results [1, 4]. Additionally, when Theorem 1 is used for half-integer \( m \), one can symbolically
integrate any given small-order case, although the structure of the closed form is obscured in the process.

While some probabilistic results such as Theorem 1 hold in all dimensions, many arithmetic and analytic results are distinct between odd and even dimensions. Indeed, even dimensional results often involve elliptic integrals [1, 2], while odd dimensional results are typically resolvable in terms of elementary functions. For instance, noting that \( j_{1/2}(x) = \text{sinc}(x) = \frac{\sin(x)}{x} \) partly explains why analysis in three-dimensional space is relatively simple. More generally, \( j_{\nu}(x) \) is elementary when \( \nu \) is a proper half-integer [1, 4, 7]. In light of this discussion, it is striking that the next result is very recent.

**Theorem 2** (Convolution formula for density in odd dimensions [3]). Assume that the dimension \( d = 2m + 1 \) is an odd number. Then for \( x \geq 0 \),

\[
p_n(m - 1/2; x) = \frac{(2x)^{2m} \Gamma(m)}{\Gamma(2m)} \left( -\frac{1}{2x} \frac{d}{dx} \right)^m P_{m,n}(x)
\]

where \( P_{m,n} \) is the piecewise polynomial obtained from convolving

\[
f_m(x) := \frac{\Gamma(m + 1/2)}{\Gamma(1/2) \Gamma(m)} \begin{cases} (1 - x^2)^{m-1} & \text{if } x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}
\]

\( n - 1 \) times with itself.

The expression in Theorem 2 above is both elegant and compact. It shows easily that in odd dimensions the density is a piecewise polynomial, but it can be difficult to manipulate or compute with or without a computer algebra system such as Maple or Mathematica. Note also that \( p_n(m - 1/2; x) = p_n(m - 1/2; -x) \) in all cases.

### 2. Main result

We now use Theorem 2 to obtain an entirely explicit and tractable, convolution and differentiation free formula for \( p_n(m - 1/2; x) \), valid for all lengths and in all odd dimensions. We begin with a preliminary result which simplifies \( P_{m,n}(x) \). We shall employ the Heaviside step function \( H(x) \) which has \( H(x) = 1 \) for \( x > 0 \), \( H(x) = 0 \) for \( x < 0 \), and \( H(0) = 1/2 \). We also use the notation \([x^j] Q(x)\) to denote the coefficient of \( x^j \) in a polynomial \( Q \).

**Proposition 3.** Let \( n \geq 1 \) and \( m \geq 1 \). Then for \( |x| \leq n \) we have \( P_{m,n}(x) = \)

\[
\left( \frac{\Gamma(2m)}{2^m \Gamma(m)} \right)^n \sum_{r=0}^{n} \binom{n}{r} (-1)^m r! \frac{H(n - 2r + x)}{(mn - 1 + j)!} \left[ x^j \right] C_m(x)^r C_m(-x)^{n-r}
\]

(2.1)

where

\[
C_m(x) := \sum_{k=0}^{m-1} \frac{(m - 1 + k)!}{2^k k!(m - 1 - k)!} x^k.
\]

(2.2)
Note that $C_m(x)$ satisfies the useful recurrence

$$C_m(x) = (2m - 3)x C_{m-1}(x) + C_{m-2}(x).$$

Moreover, in terms of hypergeometric functions $C_m(x) = \, _2F_0(m, 1 - m; \; -x/2)$.

**Proof.** By the convolution theorem for the Fourier transform,

$$\mathcal{F}(P_{m,n}(x)) = \mathcal{F}(f_m(x))^n = \left( \frac{\Gamma(m + 1/2)}{\Gamma(1/2)\Gamma(m)} \int_{-1}^{1} (1 - x^2)^{m-1} e^{-iwx} \, dx \right)^n.$$ 

Observe that, for $m \geq 3$, $\mathcal{F}(f_m(x))$ satisfies the recurrence

$$T_m = \frac{(2m-1)(2m-3)}{w^2} (T_{m-1} - T_{m-2})$$

which is also satisfied by

$$G_m(w) := \left( \frac{\Gamma(2m)}{2^m \Gamma(m)} \sum_{k=0}^{m-1} \frac{(m-1+k)!}{2^k k! (m-1-k)!} (-1)^m \frac{2 \cos(w + \frac{\pi}{2}(m+k))}{w^{m+k}} \right).$$

This can be checked by hand. It can also easily be shown with the following Maple 18 code.

```maple
with(inttrans, fourier):
f := m -> piecewise(-1 <= x and x <= 1,
                   GAMMA(m+1/2)/(GAMMA(m)*GAMMA(1/2)) * (1-x^2)^(m-1), 0):
F := m -> fourier(f(m), x, w):
simplify(F(m) - (2*m-1)*(2*m-3)/w^2*(F(m-1) - F(m-2)));
```

The above code returns 0 to indicate that $\mathcal{F}(f_m(x))$ satisfies the recurrence. Correspondingly, we may execute the following Maple 18 code.

```maple
G := m -> (GAMMA(2*m)/(2^m*m!GAMMA(m)))
       * sum((m-1+k)!/(2^k*k!*(m-1-k)!),
             k = 0..m-1)
       * (-1)^m*(2*cos(w+Pi/2*(m+k))/w^(m+k)),
       k = 0..m-1):
simplify(G(m) - (2*m-1)*(2*m-3)/w^2*(G(m-1) - G(m-2)));
```

This returns 0 to show that $G_m(x)$ satisfies the same recurrence. We can easily check that $\mathcal{F}(f_m(x))$ and $G_m(x)$ agree for $m = 1$ and $m = 2$, and so we may conclude that $\mathcal{F}(f_m(x)) = G_m(x)$ for all $m \geq 1$. 

Densities of random walks in odd dimensions 3
Therefore,

\[
\mathcal{F}(P_{m,n}(x)) = \left(\frac{\Gamma(2m)}{2^m \Gamma(m)}\right)^n \left(\sum_{k=0}^{m-1} \frac{(m-1+k)!}{2^k k!(m-1-k)!} \cdot (-1)^m \frac{2 \cos(w + \frac{\pi}{2}(m+k))}{w^{m+k}}\right)^n
\]

\[
= \left(\frac{\Gamma(2m)}{2^m \Gamma(m)}\right)^n \left(\sum_{k=0}^{m-1} \frac{(m-1+k)!}{2^k k!(m-1-k)!} \cdot (-1)^m \frac{e^{iw+i\frac{\pi}{2}(m+k)} + e^{-iw-i\frac{\pi}{2}(m+k)}}{w^{m+k}}\right)^n
\]

\[
= \left(\frac{\Gamma(2m)}{2^m \Gamma(m)}\right)^n \left(\sum_{k=0}^{m-1} \frac{(m-1+k)!}{2^k k!(m-1-k)!} \cdot (-1)^m \frac{e^{iw} + e^{-iw}}{(iw)^{m+k}}\right)^n
\]

\[
= \left(\frac{\Gamma(2m)}{2^m \Gamma(m)}\right)^n \left(\frac{e^{iw}}{iw} \right)^m C_m \left(\frac{-1}{iw}\right) + e^{-iw} \left(\frac{-1}{iw}\right) C_m \left(\frac{1}{iw}\right)^n
\]

\[
= \left(\frac{\Gamma(2m)}{2^m \Gamma(m)}\right)^n \sum_{r=0}^{n} \left(\frac{n}{r}\right) (-1)^m \sum_{j=0}^{(m-1)n} e^{i\omega(n-2r)} \frac{(n-2r+x)^{mn-j}}{(iw)^{mn+j}} [x^j] C_m(x)^r C_m(-x)^{n-r}.
\]

We can now reconstruct \(P_{m,n}(x)\) from its Fourier transform, since

\[
\mathcal{F}^{-1}\left(\frac{e^{i\omega(n-2r)}}{(iw)^{mn+j}}\right) = \frac{(n-2r+x)^{mn-1+j}}{(mn-1+j)!} H(n-2r+x) - \frac{1}{2} \frac{(n-2r+x)^{mn-1+j}}{(mn-1+j)!}.
\]

Thus, taking the inverse Fourier transform of \(\mathcal{F}(P_{m,n}(x))\),

\[
P_{m,n}(x) = \left(\frac{\Gamma(2m)}{2^m \Gamma(m)}\right)^n \sum_{r=0}^{n} \left(\frac{n}{r}\right) (-1)^m \sum_{j=0}^{(m-1)n} (n-2r+x)^{mn-j} \frac{H(n-2r+x)[x^j] C_m(x)^r C_m(-x)^{n-r}}{(mn-1+j)!}.
\]

It remains only to show that the second term above is zero. Observe that when \(x < -n\), \(P_{m,n}(x)\) simplifies to

\[
\frac{1}{2} \left(\frac{\Gamma(2m)}{2^m \Gamma(m)}\right)^n \sum_{r=0}^{n} \left(\frac{n}{r}\right) (-1)^m \sum_{j=0}^{(m-1)n} (n-2r+x)^{mn-j} \frac{H(n-2r+x)[x^j] C_m(x)^r C_m(-x)^{n-r}}{(mn-1+j)!}.
\]

From the definition of convolution, we can easily deduce that \(P_{m,n}(x)\) vanishes for \(|x| > n\). It follows that (2.4) is zero for \(x < -n\), but since it is a polynomial it must be zero everywhere. Thus, the latter term in (2.3) is zero, yielding (2.1). \(\Box\)
Next, we deal with the differential operator in Theorem 2.

**Lemma 4.** For all \( F(x) \) and \( m \geq 1 \),

\[
\left( -\frac{1}{2x} \frac{d}{dx} \right)^m F(x) = \sum_{k=1}^{m} \frac{(-1)^k(2m - 1 - k)!}{2^{2m-k}(m-k)!(k-1)!} x^{2m-k} \left( \frac{d}{dx} \right)^k F(x). \tag{2.5}
\]

**Proof.** We proceed by induction. It is trivial to see that (2.5) is true for \( m = 1 \). Suppose it holds for some \( m \geq 1 \). Then

\[
\left( -\frac{1}{2x} \frac{d}{dx} \right)^{m+1} F(x) = \left( -\frac{1}{2x} \frac{d}{dx} \right)^{m} \sum_{k=1}^{m} \frac{(-1)^k(2m - 1 - k)!}{2^{2m-k}(m-k)!(k-1)!} x^{2m-k} \left( \frac{d}{dx} \right)^k F(x)
\]

\[
= \sum_{k=1}^{m} \frac{(-1)^{k+1}(2m - 1 - k)!}{2^{2m-k+1}(m-k)!(k-1)!} \left( \frac{d}{dx} \right)^{k+1} F(x) - \frac{2m - k}{x^{2m-k+2}} \left( \frac{d}{dx} \right)^k F(x)
\]

\[
= \sum_{k=2}^{m+1} \frac{(-1)^k(2m - k)!}{2^{2m-k+2}(m + 1 - k)!(k-2)!} x^{2m-k+2} \left( \frac{d}{dx} \right)^k F(x)
\]

\[
+ \sum_{k=1}^{m} \frac{(-1)^k(2m - k)!}{2^{2m-k+1}(m-k)!(k-1)!} x^{2m-k+2} \left( \frac{d}{dx} \right)^k F(x)
\]

\[
= \sum_{k=1}^{m+1} \frac{(-1)^k(2m + 1 - k)!}{2^{2m+2-k}(m + 1 - k)!(k-1)!} x^{2m+2-k} \left( \frac{d}{dx} \right)^k F(x).
\]

Thus, (2.5) holds for all \( m \geq 1 \), proving the lemma. \( \square \)

We are now ready to approach the probability density. Combining our previous results will allow us to fully expand \( p_n(m-1/2; x) \).

**Theorem 5 (Densities in odd dimensions).** Let \( n \geq 2 \) and \( m \geq 1 \). Then for \( x \geq 0 \),

\[
p_n(m - 1/2; x) = \left( \frac{\Gamma(2m)}{2^m \Gamma(m)} \right)^n \sum_{r=0}^{n} \binom{n}{r} (-1)^r H(n - 2r + x)
\]

\[
\times \sum_{k=1}^{m} (-2)^k \binom{m - 1}{k - 1} \frac{(2m - 1 - k)!}{(2m - 1)!} x^k \sum_{j=0}^{m-1} \frac{(n - 2r + x)^{mn-1+j-k}}{(mn-1+j-k)!} [x^j] C_m(x)^r C_m(-x)^{n-r} \tag{2.6}
\]

where \( H(x) \) is the Heaviside step function and

\[
C_m(x) = \sum_{k=0}^{m-1} \frac{(m - 1 + k)!}{2^k k!(m - 1 - k)!} x^k. \tag{2.7}
\]
PROOF. By Theorem 2, Lemma 4, and Proposition 3, we arrive at

\[ p_n(m - 1/2; x) = \frac{(2x)^{2n} \Gamma(m)}{\Gamma(2m)} \left( -\frac{1}{2x} \frac{d}{dx} \right)^n P_{m,n}(x) \]

\[ = \frac{(2x)^{2n} \Gamma(m)}{\Gamma(2m)} \sum_{k=1}^{m} \frac{(-1)^k (2m - 1 - k)!}{2^{2m-k}(m - k)!(k - 1)!} \frac{1}{x^{2m-k}} \left( \frac{d}{dx} \right)^k P_{m,n}(x) \]

\[ = \left( \frac{\Gamma(2m)}{2^m \Gamma(m)} \right)^n \sum_{k=1}^{m} \frac{(-2)^k (m - 1)(2m - 1 - k)!}{k - 1} x^k \]

\[ \times \sum_{r=0}^{n} \binom{n}{r} (-1)^{mr} \sum_{j=0}^{mn-n} [x^j] C_m(x)^r C_m(-x)^{n-r} \left( \frac{d}{dx} \right)^k \frac{(n - 2r + x)^{mn-1+j}}{(mn - 1 + j)!} H(n - 2r + x). \]

We can evaluate the derivative above directly, but we must be careful since there are jump discontinuities at \( n - 2r \) for \( 0 \leq r \leq n \). We shall see that these points are not an issue. Applying the general Leibniz rule, we obtain

\[ \left( \frac{d}{dx} \right)^k \frac{(n - 2r + x)^{mn-1+j}}{(mn - 1 + j)!} H(n - 2r + x) \]

\[ = \sum_{a=0}^{k} \binom{k}{a} \left( \frac{d}{dx} \right)^a \frac{(n - 2r + x)^{mn-1+j}}{(mn - 1 + j)!} \left( \frac{d}{dx} \right)^{k-a} H(n - 2r + x) \]

\[ = \sum_{a=0}^{k} \binom{k}{a} \frac{(n - 2r + x)^{mn-1+j-a}}{(mn - 1 + j - a)!} \left( \frac{d}{dx} \right)^{k-a} H(n - 2r + x) \]

We shall see that the terms of this sum vanish except when \( a = k \). Suppose \( a < k \) and consider one such term. Clearly, \( \left( \frac{d}{dx} \right)^{k-a} H(n - 2r + x) = 0 \) for \( x \neq -n + 2r \). Additionally, since \( a < k \leq m \) and \( n \geq 2 \) the exponent \( mn - 1 + j - a \) is strictly positive, so \( (n - 2r + x)^{mn-1+j-a} = 0 \) at \( x = -n + 2r \). Thus, the summand above vanishes for \( a < k \), yielding

\[ \left( \frac{d}{dx} \right)^k \frac{(n - 2r + x)^{mn-1+j}}{(mn - 1 + j)!} H(n - 2r + x) = \frac{(n - 2r + x)^{mn-1+j-k}}{(mn - 1 + j - k)!} H(n - 2r + x) \]

We now apply this relation above and the result follows from a simple rearrangement.

\( \square \)

The formula we have presented is derived from the convolution form in Theorem 2 and produces an even function. However, \( p_n(m - 1/2; x) \) is the probability density function of a non-negative random variable, so it must be 0 for negative values of \( x \). We may use this fact to significantly reduce the number of terms in our formula, halving the time needed to compute \( p_n(m - 1/2; x) \) for given values of \( n \) and \( m \).
We finish with two examples echoing the direct analyses in [7]:

Corollary 6. Let \( n \geq 2 \) and \( m \geq 1 \). Then for \( x \geq 0 \),

\[
p_n(m - 1/2; x) = \left( \frac{\Gamma(2m)}{2^m \Gamma(m)} \right)^n \sum_{r=0}^{n} \binom{n}{r} (-1)^{mr} H(n - 2r - x) \\
\times \sum_{k=1}^{m} 2^k \binom{m-1}{k-1} (2m - 1)!(2m - 1)! x^k \sum_{j=0}^{(m-1)n} (n - 2r - x)^{mn-1+j-k} (mn - 1 + j - k)! [x^j]C_m(x)^r C_m(-x)^{n-r}
\]

(2.8)

Proof. Since our formula 2.6 is even (easily seen in Theorem 2), for \( x \geq 0 \) we have

\[
p_n(m - 1/2; x) = p_n(m - 1/2; -x)
\]

\[
= \left( \frac{\Gamma(2m)}{2^m \Gamma(m)} \right)^n \sum_{r=0}^{n} \binom{n}{r} (-1)^{mr} H(n - 2r - x) \\
\times \sum_{k=1}^{m} 2^k \binom{m-1}{k-1} (2m - 1)!(2m - 1)! x^k \sum_{j=0}^{(m-1)n} (n - 2r - x)^{mn-1+j-k} (mn - 1 + j - k)! [x^j]C_m(x)^r C_m(-x)^{n-r}
\]

by Theorem 5. Observe that when \( r > \lfloor (n - 1)/2 \rfloor \), \( H(n - 2r - x) \) is zero on \((0, \infty)\). At \( x = 0 \), every term is 0 for all values of \( r \). Thus, when \( x \geq 0 \), we may simply omit the terms where \( r > \lfloor (n - 1)/2 \rfloor \). So we let \( r \) range from 0 to \( \lfloor (n - 1)/2 \rfloor \) in the sum, which yields our result directly.

We finish with two examples echoing the direct analyses in [7]:

Example 7 (Density in three dimensions). In \( \mathbb{R}^3 \), we have \( C_1(x) = 1 \) so for \( n \geq 2 \) and \( x \geq 0 \), the density reduces to

\[
p_n(1/2; x) = \frac{-x}{2^{n-1}} \sum_{r=0}^{n} \binom{n}{r} (-1)^r H(n - 2r + x) \frac{(n - 2r + x)^{n-2}}{(n-2)!}.
\]

In particular, we have

\[
p_2(1/2; x) = \begin{cases} 
0 & \text{if } x < 0 \\
x/2 & \text{if } x \in [0, 2) \\
0 & \text{if } x > 2
\end{cases}
\]

\[
p_3(1/2; x) = \begin{cases} 
0 & \text{if } x < 0 \\
\frac{1}{4}x^2 & \text{if } x \in [0, 1) \\
-\frac{1}{4}x^2 + \frac{3}{4}x & \text{if } x \in [1, 3) \\
0 & \text{if } x > 3
\end{cases}
\]

\[
p_4(1/2; x) = \begin{cases} 
0 & \text{if } x < 0 \\
-\frac{3}{16}x^3 + \frac{1}{2}x^2 & \text{if } x \in [0, 2) \\
\frac{1}{16}x^3 - \frac{1}{2}x^2 + x & \text{if } x \in [2, 4) \\
0 & \text{if } x > 4
\end{cases}
\]
Example 8 (Density in five dimensions). In $\mathbb{R}^5$, we have $C_2(x) = 1 + x$ so for $n \geq 2$ and $x \geq 0$, the density reduces to

$$p_n(3/2; x) = \left(\frac{3}{2}\right)^{n-1} \sum_{r=0}^{n} \binom{n}{r} H(n - 2r + x) \times \sum_{j=0}^{n} \frac{(n - 2r + x)^{2n-3+j}}{(2n-3+j)!} \left(\frac{x^2}{x} - x^{(n-2r+x)} \right) \sum_{l=0}^{j} (-1)^{j-l} \binom{j}{l} \binom{n-r}{l}.$$ 

In particular, we have

$$p_2(3/2; x) = \begin{cases} 0 & \text{if } x < 0 \\ -\frac{3}{16} x^5 + \frac{3}{4} x^3 & \text{if } x \in [0, 2) \\ 0 & \text{if } x > 2 \end{cases}$$

$$p_3(3/2; x) = \begin{cases} 0 & \text{if } x < 0 \\ -\frac{3}{1120} x^8 + \frac{9}{80} x^6 - \frac{9}{32} x^4 + \frac{81}{80} x^3 - \frac{243}{1120} x & \text{if } x \in [0, 1) \\ 0 & \text{if } x > 3 \end{cases}$$

As these examples demonstrate, Theorem 5 always provides an explicit, workable expression for $p_n(m - 1/2; x)$ with clearly indicated structure. We finish by observing that since the moment function is defined by $W_n(m - 1/2, s) := \int_{x=0}^{\infty} x^s p_n(m - 1/2; x)dx$, we may also obtain an explicit formula for $W_n(m - 1/2, s)$. 

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**Figure 1. $p_n(1/2; x)$ for $n = 2, 3, 4$.**
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References


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