Probability Densities of Random Walks

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The random walk integrals

**Definition**

\[ W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right|^s \, dx \]

for complex \( s \). \( W_n := W_n(1) \).

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Let \( p_n \) be the (unique) function that satisfies

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- Work in progress...
- Makes heavy use of experimental mathematics.
What we know

- $W_1(s) = 1$, $W_2(s) = \binom{s}{s/2}$. So $p_1(x) = \delta_1(x)$, $p_2(x) = \frac{2}{\pi \sqrt{4-x^2}}$.
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- $p_n$ is unique as all moments are known and the interval of integration is finite.

- We shift focus from $W_n$ to $p_n$, in particular $p_3$ and $p_4$. 
Closed forms

**Theorem (1)**

\[
W_4(-1) = \frac{\pi}{4} \, 7F_6 \left( \begin{array}{c} 5/4, 1/2, 1/2, 1/2, 1/2, 1/2 \\ 1/4, 1, 1, 1, 1, 1 \end{array} \right| 1 \right).
\]
Closed forms

**Theorem (1)**

\[ W_4(-1) = \frac{\pi}{4} 7F_6 \left( \begin{array}{c} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 1, 1, 1, 1, 1 \end{array} | 1 \right). \]

**Theorem (2)**

*Both of the following are equal to* \( W_4(1) \):

\[
\frac{3\pi}{4} 7F_6 \left( \begin{array}{c} \frac{7}{4}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{4}, 2, 2, 2, 1, 1 \end{array} | 1 \right) - \frac{3\pi}{8} 7F_6 \left( \begin{array}{c} \frac{7}{4}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{4}, 2, 2, 2, 2, 1 \end{array} | 1 \right)
\]

\[
= \frac{9\pi}{4} 7F_6 \left( \begin{array}{c} \frac{7}{4}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{4}, 2, 2, 2, 1, 1 \end{array} | 1 \right) - 2\pi 7F_6 \left( \begin{array}{c} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 1, 1, 1, 1, 1 \end{array} | 1 \right).
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Proof of Theorem (1)

The proof uses Bailey’s identity connecting $G_{4,4}^{2,4}$ to $7\,F_6$. 

[Equation or proof not shown in the image]
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But recall that $W_4(-1)$ is a $G_{4,4}^{2,2}$. 
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Fear not! For we use the definition of Meijer G-functions to obtain the integrand for $W_4(-1)$:

$$\frac{\Gamma(\frac{1}{2} - t)^2 \Gamma(t)^2}{\Gamma(\frac{1}{2} + t)^2 \Gamma(1 - t)^2} x^t = \frac{\Gamma(\frac{1}{2} - t)^2 \Gamma(t)^4}{\Gamma(\frac{1}{2} + t)^2} \cdot \frac{\sin^2(\pi t)}{\pi^2} x^t,$$

using $\Gamma(t)\Gamma(1 - t) = \pi / \sin(\pi t)$.
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using $\Gamma(t)\Gamma(1 - t) = \pi / \sin(\pi t)$.

We choose the contour to enclose the poles of $\Gamma\left(\frac{1}{2} - t\right)$. $\sin^2(\pi t)$ does not interfere with the residues, for it equals 1 at half integers, so it can be ignored. Then the right-hand side is the integrand of a $G_{4,4}^{2,4}$. 
Proof of Theorem (2), first equality

Nesterenko’s theorem connects $G_{4,4}^{2,4}$ to a triple integral. The entries in the $G_{4,4}^{2,4}$ need to satisfy special properties. In particular,
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$$a(z) := G_{4,4}^{2,2} \left( \begin{array}{c} -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \\ 0, 1, 1, 1 \end{array} \middle| z \right)$$

does not satisfy these properties.

But $a(1) = -2\pi W_4(1)$. 

Proof of Theorem (2), first equality

Nesterenko’s theorem connects $G^{2,4}_{4,4}$ to a triple integral. The entries in the $G^{2,4}_{4,4}$ need to satisfy special properties. In particular,

$$a(z) := G^{2,2}_{4,4} \left( -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \middle| z \right)$$
does not satisfy these properties.

But $a(1) = -2\pi W_4(1)$.

However, $c := -G^{2,2}_{4,4} \left( -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \middle| 1 \right)$ does. Experimentally we observed $a(1) = 4c$. 
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*Nesterenko’s theorem* connects $G_{4,4}^{2,4}$ to a triple integral. The entries in the $G_{4,4}^{2,4}$ need to satisfy special properties. In particular, $a(z) := G_{4,4}^{2,2} \left( \frac{0,1,1,1}{-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}} | z \right)$ does not satisfy these properties.

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We use these easy identities:

$$\frac{d}{dz} \left( z^{-b_1} G_{4,4}^{2,2} \left( \frac{a_1, a_2, a_3, a_4}{b_1, b_2, b_3, b_4} \mid z \right) \right) = \frac{-1}{z^{1+b_1}} G_{4,4}^{2,2} \left( \frac{a_1, a_2, a_3, a_4}{b_1+1, b_2, b_3, b_4} \mid z \right),$$

$$\frac{d}{dz} \left( z^{1-a_1} G_{4,4}^{2,2} \left( \frac{a_1, a_2, a_3, a_4}{b_1, b_2, b_3, b_4} \mid z \right) \right) = \frac{1}{z^{a_1}} G_{4,4}^{2,2} \left( \frac{a_1-1, a_2, a_3, a_4}{b_1, b_2, b_3, b_4} \mid z \right).$$
Applying the first identity to $a(z)$ and using the *product rule*, we get $\frac{1}{2}a(1) + a'(1) = c$. 
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Using Nesterenko’s theorem:

\[
W_4(1) = \frac{4}{\pi^3} \int_0^1 \int_0^1 \int_0^1 \sqrt{\frac{x(1-y)(1-z)}{(1-x)yz(1-x(1-yz))}} \, dx \, dy \, dz.
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Using Nesterenko’s theorem:

$$W_4(1) = \frac{4}{\pi^3} \int_0^1 \int_0^1 \int_0^1 \sqrt[3]{\frac{x(1-y)(1-z)}{(1-x)yz(1-x(1-yz))}} \, dx \, dy \, dz.$$  

Change of variable $z' = 1 - z$, then use

$$(z')^{\frac{1}{2}} = (z')^{-\frac{1}{2}} (1 - (1 - z')) = (z')^{-\frac{1}{2}} - (z')^{-\frac{1}{2}} (1 - z')$$

to split it into two integrals.
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Using Nesterenko’s theorem:

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to split it into two integrals.

Each integral satisfies Zudilin’s theorem, which converts such integrals into $7F_6$'s.
Proof of Theorem (2), second equality

We convert all hypergeometric terms into triple integrals, judiciously using Bailey’s identity, Nesterenko’s theorem, and Zudilin’s theorem (for they produce multiple equivalent forms).
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These integrals are nice enough that they can be reduced to 1D integrals of $E$ and $K$.

When we pick the “right” integrals, the integrands (as functions of $E$ and $K$) on both sides equal.
For \( n \geq 4 \),

\[
p_n(t) = \int_0^\infty xt J_0(xt) J_0^n(x) \, dx
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So probability of returning to the unit disk is

$$\int_0^1 p_n(t)dt = \int_0^\infty J_1(x)J_n^n(x)dx = \left[\frac{-J_0(x)^{n+1}}{n+1}\right]_0^\infty = \frac{1}{n+1}.$$
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For $n = 2$ and 3 the probability is elementary.
For $n \geq 4$,

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For $n = 2$ and $3$ the probability is elementary.

$p_n$ is smooth for $n \geq 6$. 
• Our definition of $p_n$ takes advantage of radial symmetry. A true 2D probability density $\psi_n$ requires

$$W_n(s) = \int_0^n \psi_n(x) x^s 2\pi x \, dx.$$ 

That is, $p_n(x) = 2\pi x \psi_n(x)$. 

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Rayleigh gave approximate $\psi_n$ for large $n$, first by approximating the problem in 1D using the central limit theorem (for Bernoulli trials: $\frac{1}{\sqrt{n\pi/2}}e^{-2x^2/n}$).
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He then allowed the walks to be on a lattice, finally relaxing it to the plane, modifying his approximation.
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- $\psi_n(x) \approx \frac{1}{n\pi} e^{-x^2/n}$, like a 2D central limit theorem.
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He then allowed the walks to be on a lattice, finally relaxing it to the plane, modifying his approximation.

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like a 2D central limit theorem.

This is very accurate even for moderate $n$. 
\( p_n \) with approximations superimposed.
Recursion for $W_n$

We condition the distance $z$ of an $(n + m)$-step walk on $x$ (first $n$ steps), followed by $y$ (remaining $m$ steps).
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By the cosine rule,

$$z^2 = x^2 + y^2 + 2xy \cos(\theta).$$
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The moments are worked out by CAS:

$$g_s(x, y) := \frac{1}{\pi} \int_0^\pi z^s \, d\theta = y^s \, \text{Re} \, _2F_1 \left( \begin{array}{c} \frac{-s}{2}, \frac{-s}{2} \\ 1 \end{array} \middle| \frac{x^2}{y^2} \right).$$
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Therefore $W_{n+m}(s) = \int_0^n \int_0^m g_s(x, y) \, p_n(x)p_m(y) \, dy \, dx$.  \hspace{1cm} (1)
Recursion for $\psi_n$

Let $\mathbf{r}$ be the position vector after $n$ steps, and $\mathbf{s}$ be the position vector of the $n$th step.
Recursion for $\psi_n$

Let $r$ be the position vector after $n$ steps, and $s$ be the position vector of the $n$th step.

Then, upon using polar coordinates and the cosine rule,

$$
\psi_n(r) = \int \frac{\delta_1(|s|)}{2\pi} \psi_{n-1}(|r-s|)ds = \int_0^{2\pi} \psi_{n-1}\left(\sqrt{r^2 + 1 - 2r \cos t}\right)\frac{dt}{2\pi}.
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Combined with $\psi_2$, this gives

$$p_3(x) = \frac{\sqrt{x}}{\pi^2} \text{Re} \ K\left(\sqrt{\frac{(x + 1)^3(3 - x)}{16x}}\right).$$
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$$\psi_n(0) = \psi_{n-1}(1) = \frac{p_{n-1}(1)}{2\pi} = \frac{p'_{n}(0)}{2\pi} = \frac{\text{Res}_{-2} W_n}{2\pi}. $$
Alternative form for $p_n$

We now use the *sine rule* to make a change variable, so the last integral in (1) becomes $dz$ instead of $dx$:

$$W_{n+m}(s) = \int_0^{n+m} z^s \left\{ \int_0^n \int_0^\pi \frac{z}{\pi y} p_n(x)p_m(y) dt dx \right\} dz,$$

where $y = \sqrt{x^2 + z^2 - 2xz \cos t}$. 
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By uniqueness, the expression inside the braces is $p_{n+m}$.
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where $y = \sqrt{x^2 + z^2 - 2xz \cos t}$.

By uniqueness, the expression inside the braces is $p_{n+m}$.

Combined with $p_3$, we have

$$p_4(t) = \frac{8t}{\pi^3} \int_0^2 \text{Re} \left( \frac{K \left( \sqrt{\frac{16xt}{(x+t)^2(4-(x-t)^2)}} \right)}{\sqrt{(x+t)^2(4-(x-t)^2)}} \right) \frac{dx}{\sqrt{4-x^2}},$$

which is better numerically than its Bessel counterpart.
Poles of $W_3$ via $p_3$

In $p_3$, we have $K\left(\sqrt{\frac{16x^3}{(3-x)^3(1+x)}}\right) = \frac{3-x}{3+3x} K\left(\sqrt{\frac{16x}{(3-x)(1+x)^3}}\right)$, as both sides satisfy the same differential equation.
Poles of $W_3$ via $p_3$

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So we can write $p_3$ cleanly in terms of the AGM, enabling us to use a result of Borwein et al. So on $[0, 1)$

$$p_3(x) = \frac{2}{\sqrt{3\pi}} x \sum_{k=0}^{\infty} W_3(2k) \left( \frac{x}{3} \right)^{2k}.$$
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Using this series, we compute (with lots of care), for small $a > 0,$ 

$$\int_{0}^{a} p_3(x)x^s dx = \frac{2a^{s+2}}{\sqrt{3\pi}(s + 2)} + \frac{2a^{s+4}}{3\sqrt{3\pi}(s + 4)} + \cdots$$ 

so the residues of $W_3$ can be read off, namely,

$$\text{Res}_{(-2k-2)} W_3 = \frac{2}{\pi \sqrt{3}} \frac{W_3(2k)}{9^k}.$$

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But if $p_4$ admits a similar series, how can this reconcile with the double poles of $W_4$?
Functional equation for $p_3$

As $\text{Re } K(x) = \frac{1}{x}K\left(\frac{1}{x}\right)$ for $x > 1$, we split $p_3$ over $[0, 1]$ and $[1, 3]$, obtaining $W_3(-1) = \int_0^3 \frac{p_3(x)}{x} \, dx =$

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\frac{4}{\pi^2} \int_0^1 K\left(\sqrt{\frac{16x}{(3-x)(1+x)^3}}\right) \, dx + \frac{1}{\pi^2} \int_1^3 K\left(\sqrt{\frac{(3-x)(1+x)^3}{16x}}\right) \, dx.
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Numerically we noted the two integrals equal. Proof: change of variable $x \to \frac{3-t}{1+t}$ in the second integral.
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This leads to a modular property: with the involution $\sigma(x) = \frac{3-x}{1+x}$,

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p_3(x) = \frac{4x}{(3-x)(x+1)} p_3(\sigma(x)).
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This leads to a modular property: with the involution $\sigma(x) = \frac{3-x}{1+x}$,

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Also, $W_3(-1) = \frac{4}{\sqrt{3}\pi} \sum_{k=0}^{\infty} \frac{W_3(2k)}{9^k(2k+1)}$. 
Series for $p_4$

Jon asked us to plot $p'_4(x)$ for small $x$. Armin correctly used the true formula,

$$\lim_{h \to 0} \frac{p_4(x + h) - p_4(x)}{h},$$

I, however, foolishly used the "formula",

$$\lim_{h \to 0} \frac{p_4(x + h) - p_4(x)}{x}.$$ 

Amazingly, we produced almost the same plot, except mine was vertically translated up by $a \approx 0.14$.

Unfazed by my failure to find a derivative from first principles, this means, very nearly, $p_4$ satisfies the differential equation

$$f'(x) + ax = f(x),$$

which even I can solve:

$$f(x) = bx - ax \log x,$$

where $b \approx 0.33$ as

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This explains the double pole!
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In fact, if the series were to be consistent with the residues and coefficients of the double pole, then we must have:

\[ p_4(x) = \sum_{n=1}^{\infty} (a_4(n) - r_4(n) \log x) x^{2n-1}, \]

where \( a_4(n) \) are the residues at \(-2n\) and \( r_4(n) \) are the coefficients of the double pole at \(-2n\).
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In fact, if the series were to be consistent with the residues and coefficients of the double pole, then we must have:

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The first approximation is

$$\left( \frac{9 \log 2}{2\pi^2} - \frac{3}{2\pi^2} \log x \right) x.$$

$r_4(n)$ may be obtained in closed form by recursion.
$p_4$ versus conjectured expansion on $[0, 2]$. 
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$p_4$ can also be written in terms of the Domb numbers,

$$W_4(2n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2n - 2k}{n - k}.$$
Closed forms

From our series for $p_3$, Zudilin (using modular tools) deduced the closed form

$$p_3(x) = \frac{2\sqrt{3}x}{\pi(3 + x^2)} \; _2F_1 \left( \begin{array}{c} \frac{1}{3}, \frac{2}{3} \\ 1 \\ \end{array} \right| \frac{x^2(9 - x^2)^2}{(3 + x^2)^3} \right),$$
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as well as a closed formed for $p_4$ on $[2, 4]$:

$$p_4(x) = \frac{2\sqrt{16 - x^2}}{\pi^2 x} \ _3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \left| \frac{(16 - x^2)^3}{108x^4} \right. \right).$$

Numerically, this works on $[0, 4]$ by taking the real part.
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Numerically, this works on $[0, 4]$ by taking the real part.

We get eerie connections with $W_3(s)$, for instance

$p_4(2) = \frac{\sqrt{3}}{\pi} W_3(-1)$ and $p_3(\sqrt{3})^2 = 4p_3(2\sqrt{3} - 3)^2 = \frac{3}{2}\pi^2 W_3(-1)$. 
Future work

- Prove expansion for $p_4$, and prove closed form on all of $[0, 4]$. 
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- Links to Calabi-Yau differential equations?
- More closed forms for derivatives and residues for $W_3$ and $W_4$. 
Thank you!
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- Comments?
- Questions?
- Criticisms?